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Fluid Description of a Magnetized Toroidal Plasma

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Fluid Description of a Magnetized Toroidal Plasma

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Fluid Description of a Magnetized Toroidal Plasma

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A closed system of fluid equations to describe the evolution of a weakly collisional toroidal plasma is presented. The primary physical phenomena incorporated are gyration, guiding center motion, and parallel flows of particles and heat. The systematic use of the drift ordering allows for faster dynamics than in a transport context, and in order to capture important spatial variation, no flux-surface average is taken. A notable feature of this model is a generalized bootstrap current: applying the neoclassical limit and flux-surface average to the expression for the parallel current annihilates more general terms, leaving the canonical bootstrap current as the result. Thus, the model reduces to neoclassical transport in the proper limit, but generally includes more physical effects.

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Chapter 1

Preliminaries

The goal of this work is to provide a fluid model for plasma behavior that systematically incorporates many phenomena relevant to the magnetic confinement program which are conventionally addressed only in the framework of kinetic theory, by coarse application of its results, or neglected altogether. We carefully discuss fluid closure, starting from fundamental principles, to identify where and how these physical phenomena emerge in a fluid description. By "systematically", we mean the vigilant observance of a single ordering scheme throughout the course of calculation; in this manner, the various elements enter our model in a consistent way. The final result is a closed system of 12 partial differential equations describing the evolution of 12 fluid variables and electromagnetic field components. When the transport ordering and flux-surface averaging is applied to our system, neoclassical theory is reproduced; however, the full system includes more effects and has a greater range of validity.

Motivated by the conditions in high-temperature confinement experiments, we study the low-collisionality regime of a drift-ordered plasma. To begin our description, we introduce several conventional quantities [14]. Defining the thermal speed, $v_t \equiv \sqrt{T/m}$, we construct the frequency scale for thermal motion across a macroscopic distance scale L — the transit frequency $\omega_t \equiv v_t/L$. The drift ordering is characterized by flows and temporal evolution at the slower drift frequency. In terms of the ubiquitous small gyroparameter, δ, the ratio of the gyroradius to macroscopic length scale, the standard drift ordering can be summarized

$$\begin{aligned} \frac{\partial}{\partial t} &\sim \delta\omega_t \\ V &\sim \delta v_t \\ q &\sim \delta p v_t \end{aligned} \tag{1.1}$$

where $\omega \sim \partial/\partial t$ measures the inverse time scale for the processes of interest, V is the fluid velocity, n the density, T the temperature, and q the heat flux. Because we are interested in weakly collisional plasmas, we also order

$$\nu \ll \omega_t$$

later refining the estimate.

Addressing the complete drift-ordered problem is a daunting task; as we discuss in the next two chapters, this ordering allows many effects neglected in other orderings to enter on comparable footing. A rigorous approach to the fluid formalism[25, 27] shows considerable complexity, even prior to addressing the difficult question of closure. Our purpose here is instead to demonstrate how the particular dynamics of a toroidally confined plasma directly influence the process and results of closure. We therefore concentrate on three principle aspects: gyration, guiding center motion in toroidal fields, and parallel particle and heat flows. A system that combines all these in a systematic way will be a useful contribution to fluid theory, able to model a rich variety of plasma behavior.

1.1 Physics Overview

Among the phenomena that figure prominently in our model are: gyration driven perpendicular flows, pressure anisotropy, gyroviscosity, bootstrap current, neoclassical conductivity reduction, parallel density and temperature gradients, and parallel heat flow. Some introductory comments regarding these are appropriate here.

1.1.1 Neoclassical Effects

Neoclassical effects were originally derived under strict assumptions, appropriate to an axisymmetrically confined plasma in quasistatic equilibrium. The neoclassical fluid equations take their relatively simple form only because they have been averaged over the flux surfaces. Plasma instability, and the evolution of such symmetry breaking phenomena as magnetic islands, contradict these assumptions, depending upon effects that the averaged equations cannot show. Thus the more general dynamics require a generalized treatment. Nonetheless, if the non-equilibrium processes are slow compared to the bounce frequency of magnetically trapped particles, key neoclassical processes will continue to have important effects.

1.1.2 Parallel Flows

The literature contains a number of fluid models that include finite-Larmor-radius (FLR) as well as selected neoclassical effects[1]. The present model is distinguished from its predecessors by its emphasis on heat flow parallel to the magnetic field. At low collisionality such heat flow is rapid, but not instantaneous. Thus the parallel heat flow can play an important role in various confined plasma contexts, such as the evolution of magnetic islands and neoclassical tearing modes [6, 9, 15, 23, 31]. Furthermore, we have shown in previous work that the parallel flows lie at the root of neoclassical anisotropy and radial transport [4, 5].

[1] In chapter 3 several examples are discussed.

The transport approach to studying parallel heat flow is to express the flux $q_\|$ in terms of lower moment driving forces, such as $\boldsymbol{\nabla} T$. Results exist for arbitrary collisionality in homogeneous [12, 18] and inhomogeneous fields [13, 17]. The method presented here, however, instead promotes the parallel heat flux to a dynamical quantity which must be determined through the evolution of $\partial q_\|/\partial t$. While this increases the number of moment equations, the parallel heat is endowed with a more robust means of variation. Although the expected banana regime result is returned in the appropriate limit, in general, $q_\|$ is determined only by the full dynamics of the system of moment equations.

Unlike for the perpendicular dynamics, where the small gyroradius leads to $\mathcal{O}(\delta)$ flows, there is not a similar *a priori* restriction on the size of parallel flows. However, we do not expect them to be as large as the thermal scale and it is convenient to order them similarly to perpendicular flows

$$pV_\| \sim q_\| \sim \delta p v_t$$

One exception we wish to make is to allow for large electron parallel heat flow; we therefore employ an extended drift ordering, where all flows are first order except the electron parallel heat flux, which can be zeroth order in δ.

$$\begin{gathered} \frac{1}{\omega_t}\frac{\partial}{\partial t} \sim \frac{V}{v_t} \sim \frac{q_\perp}{pv_t} \sim \frac{q_{\|i}}{p_i v_{ti}} \sim \mathcal{O}(\delta) \\ \frac{q_{\|e}}{p_e v_{te}} \sim \mathcal{O}(1) \end{gathered} \tag{1.2}$$

1.1.3 Landau Damping

Notably absent from our discussion is a treatment of "Landau damping." Because of the significant difference between perpendicular and parallel dynamics, we allow the parallel length scale to be larger than the perpendicular length scale

$$L_\parallel \sim \frac{L_\perp}{\delta} \tag{1.3}$$

It is then consistent to have drift frequency disturbances with phase velocity equal to the thermal speed $\omega/k_\parallel \sim v_t$. At these phase velocities one has both an appreciable number of particles and significant slope of the distribution function, and so would therefore expect kinetic resonances (e.g. Landau damping effects) to be of importance.

The omission of Landau damping is a serious flaw in a model for plasma behavior at $\omega \sim kv_t$. Methods have been proposed for incorporating such kinetic effects into fluid models [10]. However, since the focus of this paper is on a particular form of moment closure and we do not yet have a consistent model for kinetic resonances in the regimes of interest, we make no attempt to incorporate Landau damping into our model and present a framework focusing on *neoclassical* effects.

1.2 Organization of Thesis

The foundation of our work is the discussion of fluid closure presented in the following chapter. After establishing the requirements for closure of the drift-ordered system of fluid equations, we take a brief look at previously published closures in Chapter 3 to provide a context for our approach. We then break up the necessary calculations into three major conceptual areas — guiding center dynamics (neoclassical effects), gyroviscosity, and collisional friction — dedicating a chapter to each. The final chapter presents the completed system of equations, and demonstrates its appropriate reduction in the neoclassical limit.

Chapter 2

Plasma as Fluid

Conventional (liquid) fluids and plasmas are both characterized by large numbers of interacting particles. The difference between short-range interactions (fluids) and long range interactions (plasmas), however, is at the heart of the particular richness of plasma physics as distinct from fluid physics. Given the sometimes drastically different behavior of the two states, it seems natural to ask in what sense plasmas can be described and analyzed as fluids. Since this issue is fundamental to much of plasma physics, it is addressed throughout the literature; I will assume some familiarity on the part of the reader and only make a few comments particularly relevant to this thesis.

2.1 Moment Hierarchy

2.1.1 Fluid Moments

The "kth moment" of the phase-space distribution function $f(\boldsymbol{x}, \boldsymbol{v})$ is the rank-k tensor defined as the velocity integral of f weighted by k factors of the velocity vector

$$\boldsymbol{M}^{(k)}(\boldsymbol{x}) \equiv \int d^3v \; m \underbrace{\boldsymbol{vvv}...\boldsymbol{v}}_{\text{k factors}} f(\boldsymbol{x}, \boldsymbol{v}) \tag{2.1}$$

In component index notation, this is written

$$M_{\alpha\beta\gamma...\omega} \equiv \int d^3v \; m v_\alpha v_\beta v_\gamma ... v_\omega f \tag{2.2}$$

These integrals are referred to as "moments", "fluid moments", or "fluid variables", interchangeably.

Familiar examples are the density

$$n \equiv \int d^3v \ f = \frac{1}{m} M^{(0)} \tag{2.3}$$

the fluid velocity

$$\boldsymbol{V} \equiv \frac{1}{n} \int d^3v \ \boldsymbol{v} f = \frac{1}{mn} \boldsymbol{M}^{(1)} \tag{2.4}$$

the (rank-2) stress tensor

$$\boldsymbol{P} \equiv \int d^3v \ m\boldsymbol{v}\boldsymbol{v} f = \boldsymbol{M}^{(2)} \tag{2.5}$$

the energy flux vector

$$\boldsymbol{Q} \equiv \int d^3v \ \frac{1}{2} m v^2 \boldsymbol{v} f = \frac{1}{2} Tr\Big(\boldsymbol{M}^{(3)}\Big) \tag{2.6}$$

and the (rank-2) energy-weighted stress tensor

$$\boldsymbol{R} \equiv \int d^3v \ \frac{1}{2} m v^2 \boldsymbol{v}\boldsymbol{v} f = \frac{1}{2} Tr\Big(\boldsymbol{M}^{(4)}\Big) \tag{2.7}$$

With the rest-frame velocity

$$\boldsymbol{v}_r \equiv \boldsymbol{v} - \boldsymbol{V} \tag{2.8}$$

we construct the rest-frame fluid moments, such as the pressure tensor

$$\boldsymbol{p} \equiv \int d^3v \ m\boldsymbol{v}_r\boldsymbol{v}_r f \tag{2.9}$$

and heat flux

$$\boldsymbol{q} \equiv \int d^3v \ \frac{1}{2} m v_r^2 \boldsymbol{v}_r f \tag{2.10}$$

Substituting for $\boldsymbol{v}_r$ gives the exact relations between lab-frame to rest-frame quantities

$$\boldsymbol{P} = \boldsymbol{p} + mn\boldsymbol{V}\boldsymbol{V} \tag{2.11}$$

and

$$\boldsymbol{Q} = \boldsymbol{q} + \boldsymbol{V} \cdot \boldsymbol{p} + \frac{3}{2} p \boldsymbol{V} + \frac{1}{2} mnV^2 \boldsymbol{V} \tag{2.12}$$

where we have defined the scalar pressure as the trace of the pressure tensor

$$p \equiv \frac{1}{3} Tr\left(\boldsymbol{p}\right) \tag{2.13}$$

Additionally, we require the energy-weighted analog to the pressure tensor

$$\boldsymbol{r} \equiv \int d^3v \ \frac{1}{2} m v_r^2 \boldsymbol{v}_r \boldsymbol{v}_r f \tag{2.14}$$

which is related to the lab-frame tensor by

$$\boldsymbol{R} = \boldsymbol{r} + \boldsymbol{V} \cdot \boldsymbol{q}^{(3)} + \boldsymbol{q}\boldsymbol{v} + \boldsymbol{v}\boldsymbol{q} + \frac{1}{2} V^2 \left(mn\boldsymbol{V}\boldsymbol{V} + \boldsymbol{p}\right) + \left(\boldsymbol{V} \cdot \boldsymbol{p}\right) \boldsymbol{V} + \boldsymbol{V} \left(\boldsymbol{V} \cdot \boldsymbol{p}\right) \tag{2.15}$$

where the rank-3 tensor $\boldsymbol{q}^{(3)}$ is the third moment of the relative velocity

$$\boldsymbol{q}^{(3)} \equiv \int d^3v \ m \boldsymbol{v}_r \boldsymbol{v}_r \boldsymbol{v}_r f$$

2.1.2 Moment Evolution

By "moment equations" ("fluid equations"), we mean those that determine the dynamical evolution of the moments

$$\frac{\partial}{\partial t} \boldsymbol{M}^{(k)} = \boldsymbol{\mathcal{F}}^{(k)}(\boldsymbol{M}) \tag{2.16}$$

with whatever particular $\boldsymbol{\mathcal{F}}^{(k)}$ describe the evolution of the plasma. Since the $\boldsymbol{M}^{(k)}$ are integrals of the distribution function f, the $\boldsymbol{\mathcal{F}}^{(k)}$ follow from appropriate integrals of the equation that determines f — the kinetic equation

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla}_x f + \frac{\mathcal{Z}e}{m} \left(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}\right) \cdot \boldsymbol{\nabla}_v f = \mathcal{C}(f) \tag{2.17}$$

Multiplying this expression by k factors of velocity and integrating results in the general moment equation [14]

$$\frac{\partial}{\partial t}\boldsymbol{M}^{(k)} + \boldsymbol{\nabla}\cdot\boldsymbol{M}^{(k+1)} - \frac{\mathcal{Z}e}{m}\left\|\boldsymbol{E}\boldsymbol{M}^{(k-1)}\right\| + \Omega\left\|\boldsymbol{b}\times\boldsymbol{M}^{(k)}\right\| = \boldsymbol{\mathcal{C}}^{(k)} \tag{2.18}$$

where the double parenthesis denotes tensor symmetrization

$$((\boldsymbol{E}\boldsymbol{A}))_{\alpha\beta\gamma\ldots\omega} \equiv E_\alpha A_{\beta\gamma\ldots\omega} + E_\beta A_{\alpha\gamma\ldots\omega} + E_\gamma A_{\beta\alpha\ldots\omega} + \ldots + E_\omega A_{\beta\gamma\ldots\alpha}$$

Ω is the gyrofrequency

$$\Omega \equiv \frac{\mathcal{Z}eB}{m}$$

and we have introduced the tensor collisional moments

$$\boldsymbol{\mathcal{C}}^{(k)}(\boldsymbol{x}) \equiv \int d^3v\ m\underbrace{\boldsymbol{v}\boldsymbol{v}\boldsymbol{v}\ldots\boldsymbol{v}}_{\text{k factors}}\ \mathcal{C}(f) \tag{2.19}$$

Specific cases of (2.18) provide familiar equations describing the fluid variables introduced above. Thus, the exact equations for the evolution of density

$$\frac{\partial n}{\partial t} + \boldsymbol{\nabla}\cdot(n\boldsymbol{V}) = 0 \tag{2.20}$$

velocity

$$mn\left(\frac{\partial\boldsymbol{V}}{\partial t} + \boldsymbol{V}\cdot\boldsymbol{\nabla}\boldsymbol{V}\right) + \boldsymbol{\nabla}\cdot\boldsymbol{p} - \mathcal{Z}en\left(\boldsymbol{E} + \boldsymbol{V}\times\boldsymbol{B}\right) = \boldsymbol{F} \tag{2.21}$$

stress tensor

$$\frac{\partial\boldsymbol{P}}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{M}^{(3)} - \mathcal{Z}en\left(\boldsymbol{V}\boldsymbol{E} + \boldsymbol{E}\boldsymbol{V}\right) + \Omega\left(\boldsymbol{b}\times\boldsymbol{P} + (\boldsymbol{b}\times\boldsymbol{P})^t\right) = \boldsymbol{C}^{(2)} \tag{2.22}$$

pressure tensor

$$\frac{3}{2}\frac{\partial p}{\partial t} + \frac{\partial}{\partial t}\left(\frac{1}{2}mnV^2\right) + \boldsymbol{\nabla}\cdot\boldsymbol{Q} = W + \boldsymbol{V}\cdot\left(\boldsymbol{F} + \mathcal{Z}en\boldsymbol{E}\right) \tag{2.23}$$

energy flux

$$\frac{\partial \boldsymbol{Q}}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{R} - \frac{3}{2}\frac{\mathcal{Z}e}{m} p\boldsymbol{E} - \frac{1}{2}\mathcal{Z}enV^2\boldsymbol{E} - \frac{\mathcal{Z}e}{m}\boldsymbol{E}\cdot\boldsymbol{P} - \frac{\mathcal{Z}e}{m}\boldsymbol{Q}\times\boldsymbol{B} = \boldsymbol{G} \tag{2.24}$$

and energy-weighted stress tensor

$$\frac{\partial \boldsymbol{R}}{\partial t} + \frac{1}{2}Tr\left(\boldsymbol{\nabla}\cdot\boldsymbol{M}^{(5)}\right) - \mathcal{Z}en\left(\boldsymbol{Q}\boldsymbol{E} + \boldsymbol{E}\boldsymbol{Q} + \boldsymbol{E}\cdot\boldsymbol{M}^{(3)}\right) + \Omega\left(\boldsymbol{b}\times\boldsymbol{R} + (\boldsymbol{b}\times\boldsymbol{R})^t\right) = \frac{1}{2}Tr\boldsymbol{C}^{(4)} \tag{2.25}$$

Here we have introduced the collisional friction

$$\boldsymbol{F} \equiv \int d^3v\ m\boldsymbol{v}\,\mathcal{C}(f) \tag{2.26}$$

collisional energy flux

$$\boldsymbol{G} \equiv \int d^3v\ \frac{1}{2}mv^2\boldsymbol{v}\,\mathcal{C}(f) \tag{2.27}$$

and collisional energy exchange

$$W \equiv \int d^3v\ \frac{1}{2}mv^2\,\mathcal{C}(f) \tag{2.28}$$

Since the parallel flows enter prominently in our formalism, we write the parallel components of (2.21) and (2.24)[1]

$$mn\left(\frac{\partial V_\parallel}{\partial t} + \boldsymbol{V}\cdot\boldsymbol{\nabla}V_\parallel\right) + \boldsymbol{b}\cdot\boldsymbol{\nabla}\cdot\boldsymbol{p} - \mathcal{Z}enE_\parallel = F_\parallel \tag{2.29}$$

and

$$\frac{\partial q_\parallel}{\partial t} + \frac{\partial}{\partial t}\left(\frac{5}{2}pV_\parallel\right) + \boldsymbol{b}\cdot\boldsymbol{\nabla}\cdot\boldsymbol{R} - \frac{\mathcal{Z}e}{m}\left[\left(\frac{3}{2}p + \frac{1}{2}mnV^2\right)E_\parallel + \boldsymbol{b}\cdot\boldsymbol{E}\cdot\boldsymbol{P}\right] = G_\parallel \tag{2.30}$$

where, for conciseness, we do not yet substitute (2.29) and (2.23) in for the time derivative of the advective heat flow.

[1]For convenience, we omit the $\partial\boldsymbol{b}/\partial t$ and $\boldsymbol{\nabla}\boldsymbol{b}$ terms, which are restored easily enough.

2.1.3 Maxwell's Equations

The fluid variables are coupled to the electric field $\boldsymbol{E}$ and magnetic field $\boldsymbol{B}$, which evolve according to Maxwell's equations. Faraday's law evolves the magnetic field

$$\frac{\partial \boldsymbol{B}}{\partial t} = -\boldsymbol{\nabla} \times \boldsymbol{E} \tag{2.31}$$

and we use the non-relativistic version of Ampere's law

$$\boldsymbol{\nabla} \times \boldsymbol{B} \cong \mu_0 \boldsymbol{J} \tag{2.32}$$

to relate the magnetic field to the current

$$\boldsymbol{J} \equiv \sum_s \mathcal{Z}_s e n_s \boldsymbol{V}_s \tag{2.33}$$

2.2 Fluid Closure

The goal of a fluid description is the evolution of a small subset of moments $\boldsymbol{\mathcal{D}} \subset \{\boldsymbol{M}^{(k)}\}$, referred to as "dynamical variables". For each dynamical variable $\mathcal{D}_i \in \boldsymbol{\mathcal{D}}$, equation (2.18) provides an evolution equation, but because of the coupling, these equations generally contain other moments not in $\boldsymbol{\mathcal{D}}$; taken as a system of equations, they cannot be solved since they refer to external unknown moments. "Closure" is the process of expressing these other moments completely in terms of the dynamical variables, since then for each dynamical variable $\mathcal{D}_i \in \boldsymbol{\mathcal{D}}$ there corresponds exactly one equation

$$\frac{\partial \mathcal{D}_i}{\partial t} = F_i(\boldsymbol{\mathcal{D}}) \tag{2.34}$$

with functions F_i of only the dynamical variables. Compare this with (2.16); the point is that, in general, the evolution of any given moment depends on all the other moments, but in a closed system, the evolution only depends on the subset of dynamical variables. Thus, the system of equations determining the evolution of $\boldsymbol{\mathcal{D}}$ becomes internally self-sufficient and complete, ready for analysis by hand or computer.

To determine the unknown moments, the fluid equations can be supplemented with a representative distribution function. Obviously, if the actual distribution function was known, even if just to some required accuracy, then the entire problem is already solved; this is, in fact, the point of kinetic theory. Fluid theory, therefore, is a tool for problems for which the full kinetic theory is too demanding. The price paid for a more tractable approach is the restriction to a model that captures certain critical features of the actual distribution function. Armed with a model distribution function parametrized by the dynamical variables

$$f(\boldsymbol{x}, \boldsymbol{v}) = f(\boldsymbol{\mathcal{D}}, \boldsymbol{v})$$

one can express all external moments in terms of those variables, closing the system of equations.

Fortunately, it turns out that a good portion of the distribution function can be determined through perturbative analysis, i.e. a rigorous approach with a specifiable accuracy. This is not completely feasible for the entire problem, however, so we then discuss an alternative option for closure, finally proposing a hybrid closure that combines both methods.

2.2.1 Perturbative Closure

Canonical fluid closure — originally formulated for gases and liquids and associated with the names Chapman, Cowling, Enskog, etc. — relies on a perturbative technique for rigorous analysis: the physical system must posses naturally identifiable small parameters for expansion and solution in power series. The smallness of the mean-free-path λ_{mfp} relative to other lengths allows such perturbation analysis for liquids, gases, and highly collisional plasmas [1]. Weakly collisional plasmas, on the other hand, do not have such a constraint on the mean-free-path; confinement plasmas may have mean-free-paths orders of magnitude larger than other length scales. If the plasma is magnetized, however, there is an alternative small parameter, namely the gyro-parameter δ. Unfortunately, the statement of small δ requires refinement and will ultimately only provide a partial solution.

The essential point is that the strong directional anisotropy introduced by the magnetic field demands a distinction between variation along versus perpendicular to the field. If collisions are infrequent, particles will traverse large distances streaming freely along the field, even though their excursions perpendicular to the field will be restricted to a small region the size of the gyroradius. Denoting the length scale perpendicular to $\boldsymbol{B}$ by $L_\perp$, the requirement that the plasma be magnetized

$$\delta_\perp \equiv \frac{\rho}{L_\perp} \ll 1 \tag{2.35}$$

implies that the perpendicular variation of the macroscopic properties of the plasma over locally gyrating orbits is small, naturally giving rise to a perturbative approach. Of course, the gyroradius must also be small compared to parallel length scale $L_\parallel$,

$$\delta_\parallel \equiv \frac{\rho}{L_\parallel} \ll 1 \tag{2.36}$$

but particles are not restricted by the field in the parallel direction, so this parameter does not describe parallel dynamics in the sense that $\delta_\perp$ does for the perpendicular directions. The conclusion is that δ can be used as a small parameter for rigorous fluid closure of the perpendicular dynamics but not the parallel dynamics. Accordingly, δ without $\perp$ or $\|$ subscript will implicitly refer to $\delta_\perp$.

2.2.2 Truncation Closure

While the condition $\delta \ll 1$ does not independently allow for a complete closure, there are useful alternatives to rigorous perturbation analysis. One approach, known as "truncation", makes use of series expansions but truncates the series with a small number of terms based on physical relevance. This has the drawback of lacking the precise knowledge of rigorous analysis, but has the benefits of being a more expedient and intuitive approximation. The following example demonstrates the general idea.

Consider expanding the velocity dependence of the distribution function

$$f = f_M \sum_{j=0}^{\infty} a_j \left(\boldsymbol{x}\right) \phi_j \left(\boldsymbol{x}, \boldsymbol{v}\right)$$

in some set of functions $\{\phi_j\}$ (orthogonal polynomials are particularly helpful). The coefficients $\{a_j\}$ are related to the fluid moments $\{\boldsymbol{M}^{(k)}\}$ as implied by (2.1) and the choice for $\{\phi_j\}$. Truncation is the process by which one argues that f is reasonably approximated, under the conditions of interest, by just a few terms. For example, to describe near-equilibrium phenomena, $\{\phi_j\}$ can be constructed so that the first term ϕ_0 corresponds to an equilibrium solution and that ϕ_1 and ϕ_2 model important departures from equilibrium. The approximate distribution function

$$f \sim \left(a_0\phi_0 + a_1\phi_1 + a_2\phi_2\right) f_M$$

would then form part of a fluid closure involving its parameters, the moments associated with $\{a_0, a_1, a_2\}$.

Another example of truncation closure is particularly familiar — MHD. Here, the distribution function is a rest-frame Maxwellian parametrized by $\{n, \boldsymbol{V}, p\}$. As in all truncation, one uses the model distribution to calculate higher order quantities (such as the results of scalar pressure and zero heat flow in MHD) in terms of the moments in f and uses the fluid equations to evolve the moments parameterizing the distribution function, e.g. evolving

$$\left\{\frac{\partial n}{\partial t}, \frac{\partial \boldsymbol{V}}{\partial t}, \frac{\partial p}{\partial t}\right\}$$

in the complete MHD system.

Notice that in contrast to a rigorous perturbative approach, truncation does not provide a prescription for judging the relative sizes of various terms, either those kept or those neglected. On the other hand, truncation can be a fast route to physical insight and involves substantially less computation when compared to a rigorous approach, by trading a model distribution function for a rigorous solution to the kinetic equation.

2.3 Hybrid Closure

Given the difficulties and benefits of various types of closure, we return to the specific physics of toroidally confined plasmas for ultimate guidance. As discussed, the smallness of δ allows a perturbative method for the perpendicular dynamics, which, according to the adopted drift ordering, is described by first order flows, cf. (1.1). Perturbative analysis of the fluid equations to properly obtain the perpendicular flows to this order is not difficult, so we therefore employ one. Lacking a natural small parameter for the parallel dynamics, we instead construct a truncation closure based on the important parallel physics: particle and heat flow. This incorporation of parallel flows is a fundamental feature of our model and deserves detailed discussion.

2.3.1 Parallel Flows as Dynamical Variables

Few fluid models for plasmas treat heat flow as a dynamical variable. Instead, various closures express the heat flow in terms of lower moments, such as density and temperature. In fact, rigorous small-δ analysis of the fluid equations shows that the lowest order *perpendicular* heat flow

$$\boldsymbol{q}_\perp = \frac{5}{2}\frac{pT}{m\Omega}\boldsymbol{b} \times \boldsymbol{\nabla} \ln T + \mathcal{O}\big(\delta^2\big) \tag{2.37}$$

has such a form[14]. The same is true for the perpendicular particle flow

$$\begin{aligned}\boldsymbol{V}_\perp &= \boldsymbol{V}_E + \boldsymbol{V}_d + \mathcal{O}\big(\delta^2\big) \\ &= \frac{\mathcal{Z}}{m\Omega}\boldsymbol{b} \times (-e\boldsymbol{E} + en\boldsymbol{\nabla} \ln p) + \mathcal{O}\big(\delta^2\big)\end{aligned} \tag{2.38}$$

In the context of closure, therefore, the perpendicular flows should not be considered independent dynamical variables since they can be calculated from the density, temperature, and electromagnetic fields.

Since the perpendicular heat flow is accurately determined by the lower moments, it is not unreasonable to expect the same for the parallel component. Several forms of such a closure are common in the literature, for example: the MHD parallel heat flow [14]

$$q_\parallel^{MHD} = 0$$

a diffusive parallel heat flow with model thermal diffusivity $\kappa_\parallel$

$$q_\parallel^d = -\kappa_\parallel \nabla_\parallel T \tag{2.39}$$

the Braginskii (large-collisionality) electron parallel heat flow [1]

$$q_\parallel^{Br} = -3.16\frac{nT\tau}{m}\nabla_\parallel T + 0.71nTV_\parallel$$

and the Hammett-Perkins (collisionless) parallel heat flow[11]

$$q_{\parallel}^{HP}(s) = -nv_t \left(\frac{2}{\pi}\right)^{3/2} \int_0^{\infty} ds' \, \frac{T(s+s') - T(s-s')}{s'} \tag{2.40}$$

where s is the distance along a field line. That these last two limits can be unified into a single expression for all collisionalities has been recognized by several authors [12, 18], leading to expressions of the form

$$q_{\parallel}^{un}(s) = nv_t \left(\frac{2}{\pi}\right)^{3/2} \int ds \, \mathcal{K}(s) \, \frac{T(s+s') - T(s-s')}{s'}$$

which reproduce both (2.39) and (2.40) in the appropriate limit.

Each of these examples, with (2.37), leads to a heat flow determined by the lower moments (remember that $\boldsymbol{V}_{\perp}$ and $\boldsymbol{q}_{\perp}$ are determined by $\{n, p, \boldsymbol{E}, \boldsymbol{B}\}$)

$$q_{\parallel} = q_{\parallel}\big(n, V_{\parallel}, p, \boldsymbol{E}, \boldsymbol{B}\big)$$

The divergence of such an expression closes the moment equation hierarchy at (2.23). In this case, no moment equations beyond $\partial p/\partial t$ are needed and the set of dynamical variables is essentially $\{n, V_{\parallel}, p, \boldsymbol{E}, \boldsymbol{B}\}$.

Recall, however, that the Braginskii system and diffusive forms (2.39) are only appropriate in the large-collisionality regime, where the smallness of the mean-free-path restricts heat flow to be a relatively local phenomenon (gradient). This limitation was a major motivation for the above studies of heat flow in weak or arbitrary collisionalities. Notice that as the collisionality is reduced, allowing parallel streaming over larger distances, the heat flow becomes highly non-local, a complicated global integral of the temperature variation along a field line.

An alternative approach is to instead promote the parallel heat flow to the status of a full dynamical variable. As such, $q_{\|}$ is not merely calculated from the lower moments; a closed system of equations must now include (2.30), with $q_{\|}$ given fully only by the solution of that system. In other words, the set of fluid dynamical variables becomes

$$\boldsymbol{\mathcal{D}} = \{n, V_{\|}, p, q_{\|}, \boldsymbol{E}, \boldsymbol{B}\} \tag{2.41}$$

each of which is determined by solution of the coupled fluid equations

$$\left\{ \frac{\partial n}{\partial t}, \frac{\partial V_{\|}}{\partial t}, \frac{\partial p}{\partial t}, \frac{\partial q_{\|}}{\partial t} \right\}$$

and Maxwell's equations.

2.3.2 Parallel Flows and Truncation

In addition to requiring (2.30) in a candidate system of equations, treating the parallel heat flow as a dynamical variable leads to a natural truncation closure for the parallel dynamics. Recall from the discussion of Section 2.2.2 that truncation is based on constructing a distribution function that models the intended physics. This amounts to choosing a reasonable distribution function that *allows* the system to have parallel flows of particles and heat, but does not constrain or determine them, which is instead the purpose of the fluid equations (2.29) and (2.30). The terms intended to model the parallel flows will be parametrized by the moments $V_{\|}$ and $q_{\|}$ without reference to higher moments, thus providing closure for the parallel dynamics.

In making this procedure more explicit, it is convenient to consider describing moments in terms of operators which will simplify the conceptual separation of various parts of the distribution function. For example, given any phase-space function $\phi(\boldsymbol{x}, \boldsymbol{z})$ we can construct a density operator which is naturally given by

$$n[\phi] \equiv \int d^3v \ \phi$$

The true physical density is $n[f]$ where f is the exact distribution function, but we can also ask whether arbitrary functions have a density; for example, the result

$$n\left[\exp\left(-\frac{mv^2}{2T}\right)\right] = \left(\frac{2\pi T}{m}\right)^{\frac{3}{2}}$$

determines precisely the normalization of the Maxwellian distribution function

$$f_M \equiv n\left(\frac{m}{2\pi T}\right)^{3/2} \exp\left(-\frac{mv^2}{2T}\right) \tag{2.42}$$

and the null result

$$n\left[v_{\|}g(v)\right] = 0$$

for any function g of the magnitude v shows that functions that are odd in parallel velocity do not contribute to the density. Similarly, one can construct the parallel heat flow operator

$$q_{\|}[\phi] \equiv \boldsymbol{b} \cdot \int d^3v \ \frac{m}{2} v_r^2 \boldsymbol{v}_r \phi$$

which, when operating on the true distribution function f, returns exactly the total parallel heat flow.

To effect a truncation closure based on parallel flows, we introduce the model distribution function

$$f_{\|} \equiv f_{\|v} + f_{\|q} \tag{2.43}$$

where the each term is designed to represent only one kind of flow, with no contribution to the other moments. In other words, while neither has density or pressure

$$\begin{aligned} n[f_{\|v}] &= n[f_{\|q}] = 0 \\ p[f_{\|v}] &= p[f_{\|q}] = 0 \end{aligned} \tag{2.44}$$

one term supplies the correct particle flow

$$\begin{aligned} V_{\|}[f_{\|v}] &= V_{\|} \\ q_{\|}[f_{\|v}] &= 0 \end{aligned} \tag{2.45}$$

and the other supplies the correct heat flow

$$\begin{aligned} V_{\|}[f_{\|q}] &= 0 \\ q_{\|}[f_{\|q}] &= q_{\|} \end{aligned} \tag{2.46}$$

Here we introduce the operators, cf. (2.5)-(2.13)

$$p[\phi] \equiv \frac{1}{3} \int d^3v \; mv_r^2 \phi$$

and

$$V_{\|}[\phi] \equiv \int d^3v \; v_{\|} \phi$$

The explicit expressions

$$\begin{aligned} f_{\|} &\equiv f_{\|v} + f_{\|q} \\ f_{\|v} &\equiv -\frac{m}{T} V_{\|} v_{\|} f_M \\ f_{\|q} &\equiv -\frac{2m}{5pT} \left(\frac{mv^2}{2T} - \frac{5}{2} \right) q_{\|} v_{\|} f_M \end{aligned} \tag{2.47}$$

are simple forms that satisfy the above criteria (2.44)-(2.46).

Notice that the framework of orthogonal functions, especially Legendre and Laguerre polynomials, would offer a particularly expedient description of the preceding discussion. As we have so far not needed more than one or two terms, however, we will postpone introducing such methods for now.

2.3.3 Distribution Function

In contrast to the parts of the distribution function that require modeling for truncation closure, the gyrophase dependent corrections can be derived rigorously[14]. A distribution function with gyroaverage $\overline{f}$ has the gyrophase dependent correction

$$\widetilde{f} = -\boldsymbol{\rho} \cdot \left[\boldsymbol{\nabla} \overline{f} + e\boldsymbol{b} \times \boldsymbol{v}_D \frac{\partial \overline{f}}{\partial \mu} - e \frac{\partial \boldsymbol{A}}{\partial t} \frac{\partial \overline{f}}{\partial U} \right] + \frac{v_\parallel \mu}{\Omega} \frac{\partial \overline{f}}{\partial \mu} \left(\frac{\boldsymbol{\rho} \boldsymbol{v}_\perp}{\rho v_\perp} : \boldsymbol{\nabla} \boldsymbol{b} - \frac{1}{2} \boldsymbol{b} \cdot \boldsymbol{\nabla} \times \boldsymbol{b} \right) \tag{2.48}$$

where the derivatives are independent partial derivatives in $(\boldsymbol{x}, \mu, U)$ phase space; in other words, the gradient is taken at constant magnetic moment μ and total energy U, $\partial/\partial\mu$ at constant position $\boldsymbol{x}$ and U, and $\partial/\partial U$ at constant $\boldsymbol{x}$ and μ. This formidable looking expression simplifies considerably to a familiar form when the plasma of interest is close to a Maxwellian $f = f_M + \mathcal{O}(\delta)$; in this case, then the lowest order gyro-correction becomes

$$\widetilde{f} = -\boldsymbol{\rho} \cdot \boldsymbol{\nabla} f_M + \frac{e}{T} \boldsymbol{\rho} \cdot \frac{\partial \boldsymbol{A}}{\partial t} f_M + \mathcal{O}\big(\delta^2\big)$$

The second term, which may not look as familiar as the first, is merely how the electrodynamic component of the electric field $\boldsymbol{E}_A \equiv \partial \boldsymbol{A}/\partial t$ enters to combine with the electrostatic field contribution $\boldsymbol{E}_\phi = -\boldsymbol{\nabla}\phi$ from the Maxwellian

$$\boldsymbol{\nabla} f_M = f_M \left[\boldsymbol{\nabla} \ln n + \left(\frac{mv^2}{2T} - \frac{3}{2} \right) \boldsymbol{\nabla} \ln T + \frac{e}{T} \boldsymbol{\nabla} \phi \right] \tag{2.49}$$

so that

$$\widetilde{f} = -\boldsymbol{\rho} \cdot \left[\boldsymbol{\nabla} \ln n + \left(\frac{mv^2}{2T} - \frac{3}{2} \right) \boldsymbol{\nabla} \ln T - \frac{e\boldsymbol{E}}{T} \right] f_M + \mathcal{O}\big(\delta^2\big) \tag{2.50}$$

where the full electric field appears. However, in the drift ordering, the electric field is primarily electrostatic [14]

$$\boldsymbol{E} = -\boldsymbol{\nabla}\phi + \mathcal{O}(\delta)$$

and it is sufficient to use

$$\widetilde{f} = -\boldsymbol{\rho} \cdot \boldsymbol{\nabla} f_M + \mathcal{O}\big(\delta^2\big) \tag{2.51}$$

or, equivalently, (2.50) with an electrostatic field.

Equation (2.51), which generates a great deal of familiar FLR physics, is first order in δ. We will soon show, however, that evolving first order quantities such as $\boldsymbol{V}$ and $\boldsymbol{q}$ requires certain $\mathcal{O}(\delta^2)$ corrections. Fortunately, these can be acquired through manipulation of the fluid equations and use of only $\mathcal{O}(\delta)$ corrections to f rather than using any second order corrections. Thus, we explicitly write the full form of the distribution function as

$$\begin{aligned} f &= \overline{f} + \widetilde{f} \\ \widetilde{f} &= -\boldsymbol{\rho} \cdot \boldsymbol{\nabla} f_M + \mathcal{O}(\delta^2) \\ \overline{f} &= f_M + f_{\|} + f_b + \mathcal{O}(\delta^2) \end{aligned} \tag{2.52}$$

(we will define f_b presently) but it must be kept in mind that the unwritten higher-order corrections are not completely inaccessible and will enter our formulation implicitly.

Because the gyro-dependent part of the distribution function $\widetilde{f}$ is a straightforward and rigorous calculation given the gyro-averaged part, the real essence of the distribution function is the form for $\overline{f}$. Equation (2.52) shows three terms. The largest part is the Maxwellian (parametrized by the *total* density and pressure)

$$\overline{f}_0 \equiv f_M \sim \mathcal{O}(1) \tag{2.53}$$

meaning that the plasma is "close" to equilibrium, the size of the departures dictated by the first order corrections

$$\overline{f}_1 \equiv f_b + f_{\|} \sim \mathcal{O}(\delta) \tag{2.54}$$

Chapter 4 contains a detailed discussion of the justifications for and derivation of this distribution function. Here we only mention that the purpose of f_b is to ensure that the physics of banana regime plasmas is captured by our model. We include $f_{\|}$ since we must also go beyond conventional neoclassical theory, in particular, for our treatment of parallel particle and heat flow. Both $\widetilde{f}$ and f_b (as will be shown) are perturbative solutions to kinetic equations and represent rigorous, if approximate, dynamics; by contrast, $f_{\|}$, given by (2.43) and (2.47), is a truncation closure necessitated by the non-local nature of the parallel dynamics in a weakly collisional plasma.

With a form for f, closure follows from the systematic application of the drift ordering to the fluid equations and the use of this model distribution function when necessary.

2.4 Complete System

Having established the fundamental set of dynamical fluid variables (2.41), we return to the moment equations to complete the process of fluid closure. At this point, it is beneficial to recall that the system of interest is a drift-ordered plasma; the exact equations can be simplified by the systematic application of the orderings (1.1). We next apply this drift ordering and identify the final calculations necessary for a closed system of equations. For these "closure requirements", the specifics of confinement plasmas determines what physical phenomena must be investigated in the following chapters — guiding center motion in toroidal magnetic fields, finite-Larmor-radius flows and stresses, and weak, but important, collisional effects.

2.4.1 Ordered Equations

The essence of the drift ordering is moderation: spatial and temporal variation is neither very fast, nor very slow, relative to thermal scales. Thus, in a hierarchy of orderings, the drift regime

$$\frac{\partial}{\partial t} \sim \delta\omega_t$$

$$V \sim \delta v_t$$

lies squarely between the rapid dynamics of MHD

$$\left.\frac{\partial}{\partial t}\right|_{MHD} \sim \omega_t$$

$$V_{MHD} \sim v_t$$

and the slow evolution of a transport (neoclassical) plasma

$$\left.\frac{\partial}{\partial t}\right|_{NC} \sim \delta^2 \omega_t$$

$$\langle V_r \rangle \sim \delta^2 v_t$$

Such a middle ground provides a relatively egalitarian arena for competing physical effects. In contrast to MHD, there are no terms ordered dominant to the exclusion of all others, and relative to transport, many otherwise neglected terms now contribute significantly. In the same spirit, we order collisions so that, while appropriately weak for high-temperature plasmas, they are neither dominant nor negligible relative to many other terms by using

$$\nu \sim \omega \sim \delta \omega_t \tag{2.55}$$

The drift-ordered fluid equations are then derived by applying (2.4.1) and (2.55) systematically to the exact equations, (2.20), (2.21), (2.22), and (2.24). Since our model is designed to allow for large parallel electron heat flow, however, we make use of the extended drift ordering discussed in Section 1.1.2. In any case, the critical point is the *systematic* application of the orderings. For example, our use of $V_\parallel \sim \delta v_t$ and $\partial/\partial t \sim \delta \omega_t$ implies

$$\frac{\partial V_\parallel}{\partial t} \sim \delta^2 \omega_t v_t \sim \mathcal{O}\big(\delta^2\big)$$

Therefore, $\mathcal{O}\big(\delta^2\big)$ accuracy must be kept in all the other terms in (2.29) to properly evolve $V_\parallel$.

Before analyzing the fluid equations, we first examine the difference between rest-frame and lab-frame quantities; the smallness of $V \sim \mathcal{O}(\delta)$ reduces the advective terms in the stress tensor, (2.11), which becomes

$$\boldsymbol{P} = \boldsymbol{p} + \mathcal{O}\big(\delta^2\big) \tag{2.56}$$

and the energy flow, (2.10), which becomes

$$\boldsymbol{Q} = \boldsymbol{q} + \boldsymbol{p} \cdot \boldsymbol{V} + \frac{3}{2} p \boldsymbol{V} + \mathcal{O}\big(\delta^2\big) \tag{2.57}$$

A further simplification occurs because the distribution function is nearly isotropic. By construction, the Maxwellian in (2.52) contributes the entire scalar pressure

$$p[f] = p[f_M] = p \sim \mathcal{O}(1)$$

and all other parts of the distribution function have small contribution to the pressure tensor

$$\boldsymbol{p}[f - f_M] \sim \mathcal{O}(\delta)$$

Thus, the pressure tensor is approximately diagonal

$$\boldsymbol{p} = p\boldsymbol{I} + \mathcal{O}(\delta) \tag{2.58}$$

This type of reduction follows naturally from the confined plasmas our model is intended to describe: they are not far from equilibrium, but those small deviations are interesting and important. Although a careful calculation of higher order corrections to the pressure tensor will occupy a substantial portion of later sections, here we see that only the isotropic part contributes to first order in (2.57), which becomes

$$\boldsymbol{Q} = \boldsymbol{q} + \frac{5}{2} p \boldsymbol{V} + \mathcal{O}\big(\delta^2\big) \tag{2.59}$$

2.4.1.1 Density and Pressure Evolution

The continuity equation is consistent as it stands in (2.20) with all terms $\mathcal{O}(\delta)$. In the evolution of the scalar pressure (2.23), it is accurate to neglect $\mathcal{O}(\delta^2)$, so that

$$\frac{3}{2}\frac{dp}{dt} + \boldsymbol{\nabla}\cdot\boldsymbol{q} + \frac{5}{2}p\boldsymbol{\nabla}\cdot\boldsymbol{V} = \boldsymbol{V}\cdot(\boldsymbol{F} + \mathcal{Z}en\boldsymbol{E} - \boldsymbol{\nabla}p)$$

where we have used (2.58), (2.59) and

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \boldsymbol{V}\cdot\boldsymbol{\nabla}$$

for the full convective derivative and neglected the energy exchange W as small in the mass ratio. The right-hand-side, from (2.21) and (2.58), can be seen to nearly cancel

$$\boldsymbol{V}\cdot(\boldsymbol{F} + \mathcal{Z}en\boldsymbol{E} - \boldsymbol{\nabla}p) \sim \mathcal{O}(\delta^2)$$

so that the evolution of the scalar pressure is given by

$$\frac{3}{2}\frac{dp}{dt} + \boldsymbol{\nabla}\cdot\boldsymbol{q} + \frac{5}{2}p\boldsymbol{\nabla}\cdot\boldsymbol{V} = 0 \tag{2.60}$$

which, like the continuity equation, already only refers to dynamical variables.

2.4.1.2 Momentum Evolution

Recall that in the context of small gyro-parameter $\delta \ll 1$, the momentum evolution equation, (2.21), perturbatively determines the perpendicular flow. Taking the cross-product gives

$$n\boldsymbol{V}_\perp = \frac{1}{m\Omega}\boldsymbol{b}\times\left[mn\frac{d\boldsymbol{V}}{dt} + \boldsymbol{\nabla}\cdot\boldsymbol{p} - \mathcal{Z}en\boldsymbol{E} - \boldsymbol{F}\right]$$

which shows [14] that the lowest order perpendicular velocity is the sum of the $\boldsymbol{E} \times \boldsymbol{B}$ and diamagnetic drifts

$$
\begin{aligned}
n\boldsymbol{V}_{\perp} &= n\boldsymbol{V}_{E} + n\boldsymbol{V}_{d} + \mathcal{O}\big(\delta^{2}\big) \\
\boldsymbol{V}_{E} &\equiv \frac{\mathcal{Z}e}{m\Omega}\boldsymbol{E} \times \boldsymbol{b} \\
\boldsymbol{V}_{d} &\equiv \frac{T}{m\Omega}\boldsymbol{b} \times \boldsymbol{\nabla} \ln p
\end{aligned}
\tag{2.61}
$$

That the perpendicular dynamics can be rigorously determined perturbatively using $\delta \ll 1$ has already been alluded to; one can easily show that the same result obtains using the perturbative solution

$$
\widetilde{f} \sim -\boldsymbol{\rho} \cdot \boldsymbol{\nabla} f_{M} \sim \mathcal{O}(\delta)
$$

in the defining integral (2.4). Equation (2.61) is sufficiently accurate to determine the drift-ordered $\boldsymbol{V}_{\perp} \sim \mathcal{O}(\delta)$ throughout the model, but since the approximate Ampere's law (2.32) neglects temporal derivatives of the electric field, (2.21) must be used to evolve the perpendicular electric field through the chain of relations

$$
\frac{d\boldsymbol{V}}{dt} \rightarrow \frac{d\boldsymbol{V}_{\perp}}{dt} \rightarrow \frac{d\boldsymbol{V}_{E}}{dt} \rightarrow \frac{d\boldsymbol{E}}{dt}
$$

To this end, we will use the perpendicular components of (2.21),

$$
mn\frac{d\boldsymbol{V}_{E}}{dt} + mn\frac{d\boldsymbol{V}_{d}}{dt} + (\boldsymbol{\nabla} \cdot \boldsymbol{p})_{\perp} - \mathcal{Z}en\left(\boldsymbol{E}_{\perp} + \boldsymbol{V} \times \boldsymbol{B}\right) = \boldsymbol{F}_{\perp} \tag{2.62}
$$

as an implicit form for $\partial \boldsymbol{E}_{\perp}/\partial t$.

The parallel component (2.29) of the momentum equation needs no modification

$$
mn\frac{dV_{\|}}{dt} + \boldsymbol{b} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{p} - \mathcal{Z}enE_{\|} = F_{\|}
$$

As we demonstrate in Section 2.4.2, because of the small mass-ratio, the ion version of this equation will be used to evolve the ion parallel flow $V_{\| i}$ and the electron equation, on account of the negligible electron inertia, instead determines $E_{\|}$.

For each of these, as with (2.62), a critical feature is the second order accuracy required by

$$\frac{\partial V}{\partial t} \sim \delta^2 \omega_t v_t$$

Therefore, the divergence of the pressure tensor $\boldsymbol{\nabla} \cdot \boldsymbol{p}$ and the friction force $\boldsymbol{F}$, which are not dynamical variables, must be calculated, i.e. expressed in terms of dynamical variables, to this same order accuracy. The bulk of the calculation required for our closure will be these quantities and their higher order analogs, $\boldsymbol{\nabla} \cdot \boldsymbol{R}$ and $\boldsymbol{G}$, which appear in the evolution of the heat flow, presently.

2.4.1.3 Heat Flow Evolution

Equation (2.24) for the evolution of energy flow has the same structure as the momentum evolution equation (2.21) and can be manipulated in an analogous way to extract the lowest order perpendicular heat flow [14]

$$\boldsymbol{q}_\perp = \frac{5}{2}\frac{pT}{m\Omega}\boldsymbol{b} \times \boldsymbol{\nabla} \ln T + \mathcal{O}\big(\delta^2\big)$$

As the was the case for the perpendicular flow, this is sufficient to determine $\boldsymbol{q}_\perp$ to the necessary $\mathcal{O}(\delta)$ accuracy. Turning to the evolution of parallel heat flow, we insert (2.23) and (2.29) into (2.30)

$$\begin{aligned}\frac{\partial q_\parallel}{\partial t} + \boldsymbol{b} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{R} - \frac{5}{2}\frac{p}{mn}\boldsymbol{b} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{p} + \frac{\mathcal{Z}e}{m}\left(pE_\parallel - \boldsymbol{b} \cdot \boldsymbol{E} \cdot \boldsymbol{p}\right) \\ -V_\parallel \boldsymbol{V}_\perp \cdot \boldsymbol{\nabla} p - \frac{5}{2}p\boldsymbol{V} \cdot \boldsymbol{\nabla} V_\parallel - \frac{5}{2}V_\parallel \boldsymbol{V} \cdot \boldsymbol{\nabla} p \\ -\frac{5}{3}V_\parallel\left(\boldsymbol{\nabla} \cdot \boldsymbol{q} + \frac{5}{2}p\boldsymbol{\nabla} \cdot \boldsymbol{V}\right) = G_\parallel - \frac{5}{2}\frac{p}{mn}F_\parallel\end{aligned}$$

assuming $E_\parallel$ is not very large, and neglecting $\mathcal{O}\big(\delta^3\big)$ throughout. We have already seen in (2.58) that the largest part of the pressure tensor is the scalar component. The same arguments apply to the energy-weighted stress tensor $\boldsymbol{R}$. Therefore, we define the non-scalar parts of $\boldsymbol{p}$ and $\boldsymbol{R}$ as the traceless tensors

$$\boldsymbol{p}^{ns} \equiv \boldsymbol{p} - p\boldsymbol{I}$$

$$\boldsymbol{R}^{ns} \equiv \boldsymbol{R} - R\boldsymbol{I}$$

$$R \equiv \frac{1}{3} Tr\left(\boldsymbol{R}\right)$$

Both of these will be small by at least one power of δ

$$p^{ns} \sim \delta p$$

$$R^{ns} \sim \delta R$$

but in fact, must be calculated to $\mathcal{O}\big(\delta^2\big)$ for use in the parallel flow and heat flow equations. Since the only contribution to the scalars comes from the Maxwellian,

$$\begin{aligned} R[f] = R[f_M] &= \frac{5}{2}\frac{pT}{m} + \frac{1}{2}V^2\left(mnV^2 + 5p\right) \\ &\sim \frac{5}{2}\frac{pT}{m} + \frac{5}{2}pV^2 + \mathcal{O}\big(\delta^4\big) \end{aligned}$$

giving the parallel heat flow equation

$$\begin{aligned} \frac{\partial q_\parallel}{\partial t} + \frac{5}{2}\frac{p}{m}\nabla_\parallel T + \frac{5}{2}\nabla_\parallel\left(pV^2\right) + \boldsymbol{b}\cdot\boldsymbol{\nabla}\cdot\boldsymbol{R}^{ns} - \frac{5}{2}\frac{p}{mn}\boldsymbol{b}\cdot\boldsymbol{\nabla}\cdot\boldsymbol{p}^{ns} - \frac{\mathcal{Z}e}{m}\boldsymbol{E}\cdot\boldsymbol{b}\cdot\boldsymbol{p}^{ns} \\ -V_\parallel \boldsymbol{V}_\perp\cdot\boldsymbol{\nabla}p - \frac{5}{2}\boldsymbol{\nabla}\cdot\big(pV_\parallel\boldsymbol{V}\big) - \frac{5}{3}V_\parallel\left(\boldsymbol{\nabla}\cdot\boldsymbol{q} + \frac{5}{2}p\boldsymbol{\nabla}\cdot\boldsymbol{V}\right) = G_\parallel - \frac{5}{2}\frac{p}{mn}F_\parallel \end{aligned} \tag{2.63}$$

Every term in this equation is properly expressed in terms of the dynamical variables (2.41) except $\boldsymbol{p}^{ns}$, $\boldsymbol{R}^{ns}$, $F_\parallel$, and $G_\parallel$, which each must be determined to $\mathcal{O}\big(\delta^2\big)$. Here it is clear that the choice (2.55) orders the collisional terms comparable to the others, since they will contain one power of δ from the small collision frequency and another from the fact that only departures from a Maxwellian (which are small) contribute to the collisional frictions

$$F_\parallel[f] = F_\parallel[f - f_M] \sim mv_t\nu\delta n \sim \delta^2\omega_t mnv_t$$

$$G_\parallel[f] = G_\parallel[f - f_M] \sim Tv_t\nu\delta n \sim \delta^2\omega_t pv_t$$

We have now established the main equations for evolving the fundamental dynamical variables. However, we have not yet taken much advantage of another very important small factor, the ratio of the electron to ion mass, a topic we turn to next.

2.4.2 Mass Discrepancy: $m_i \gg m_e$

There are numerous places in plasma physics where the difference of three orders of magnitude between the ion and electron masses leads to useful, often substantial, simplification. We refer to the small mass-ratio frequently throughout this work; here we discuss how the mass discrepancy alters the relative importance of different fluid variables, allowing some to be neglected, such as electron inertia, and others, such as electron parallel heat flow, to be quite large.

Equation (2.17) shows, through the Lorentz acceleration, that the fundamental dynamical property of a charged particle is the charge-to-mass-ratio $\mathcal{Z}e/m$. However, the possible values for $\mathcal{Z}$ are not too large, especially for the low-$\mathcal{Z}$ fuels for fusion, so the significant factor for comparing ions and electrons is the mass. This is clearly the case for hydrogenic ions ($\mathcal{Z}_H = 1$), which we will focus on for simplicity; the modification to include other ions is a straightforward task.

With such a small mass relative to ions, electrons have much less resistance to motion. Thus, in the same magnetic field, electrons have a substantially larger perpendicular acceleration, gyrating much more rapidly

$$\frac{\Omega_e}{\Omega_i} = \frac{m_i}{m_e}$$

over gyro-orbits of smaller extent (given comparable temperatures $T_e \sim T_i$)

$$\frac{\rho_e}{\rho_i} \sim \sqrt{\frac{m_e}{m_i}}$$

In the parallel direction, the electrons are again easily accelerated (but not by the magnetic field); one speaks of “free-streaming” along the field lines and might expect large fluxes for the lightweight electrons. If the parallel electron flux greatly exceeded the ion flux, however, a substantial parallel current would be generated through (2.33); that this is not observed implies that the electron and ion parallel velocities are comparable. Then the inertial acceleration term is much larger for the ions

$$\begin{aligned} m_i \frac{dV_{\|i}}{dt} &\sim m_i \delta_i \omega_{ti} V_{\|} \\ &\sim m_i \delta_e \omega_{te} V_{\|} \\ &\sim \left(\frac{m_i}{m_e}\right) m_e \delta_e \omega_{te} V_{\|} \\ &\sim \left(\frac{m_i}{m_e}\right) m_e \frac{dV_{\|e}}{dt} \end{aligned} \tag{2.64}$$

since the product $\delta\omega_t$ is independent of mass. It is then useful to consider the sum of the electron and ion parallel momentum equations, neglecting electron inertia

$$m_i n \frac{dV_{\|i}}{dt} + \nabla_{\|} \left(p_i + p_e\right) + \boldsymbol{b} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{p}_i^{ns} + \boldsymbol{b} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{p}_e^{ns} = 0 \tag{2.65}$$

where the electric field terms have canceled and $F_{\|e} = -F_{\|i}$ is required by collisional momentum conservation in a single species plasma.

Now consider the electron version of (2.29). For comparable temperatures $p_e \sim p_i$ and the parallel gradients for the two species are comparable. The parallel electric and friction forces both also appear in the ion equation, implying that they are comparable, if not larger, than the ion inertial term. Thus every term other than $dV_{\|e}/dt$ is ordered by the ion scale, which (2.64) shows to be much greater than electron inertia. Neglecting this small term gives

$$\nabla_{\parallel} p_e + \boldsymbol{b} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{p}_e^{ns} + enE_{\parallel} = F_{\parallel e} \tag{2.66}$$

as a parallel "Ohm's law". Physically, the electron acceleration is so rapid that $V_{\parallel e}$ responds to the ion flow speed on a faster time scale than the other parallel dynamics, remaining approximately constant and absent in (2.66). Since $V_{\parallel e}$ can now be determined from the current (magnetic field) and ion parallel velocity

$$V_{\parallel e} = V_{\parallel i} - \frac{J_{\parallel}}{ne} = V_{\parallel i} - \frac{1}{ne} \boldsymbol{b} \cdot \boldsymbol{\nabla} \times \boldsymbol{B} \tag{2.67}$$

it is no longer necessary to treat it as a separate dynamical variable. Equation (2.66) instead provides an algebraic expression for the parallel electric field, contributing one more equation for closure.

Quite a different situation occurs regarding parallel heat flow. Ambipolarity, such as in the above discussion, and quasi-neutrality are examples of mechanisms that constrain charged particle flows. No such obvious analogues are present to impede the parallel flow of heat. The thermal flow scale

$$q_t \sim p v_t$$

is likely to be the upper limit, and shows that for comparable temperatures, electron heat flow would be dominant. While we expect ion heat flow to be accurately described instead by the drift ordering

$$q_{\parallel i} \sim \delta_i p_i v_{ti}$$

we extend the drift ordering to *allow* for large electron parallel heat flow

$$q_{\parallel e} \sim p_e v_{te}$$

In this case, (2.63) simplifies somewhat since heat flow dominates simple advection

$$q_{\|e} \gg p_e V_e \sim \delta_e p_e v_{te}$$

leaving

$$\begin{aligned} \frac{\partial q_{\|e}}{\partial t} + \frac{5}{2}\frac{p_e}{m_e}\nabla_\| T_e + \boldsymbol{b}\cdot\boldsymbol{\nabla}\cdot\boldsymbol{R}_e^{ns} - \frac{5}{2}\frac{p_e}{m_e n}\boldsymbol{b}\cdot\boldsymbol{\nabla}\cdot\boldsymbol{p}_e^{ns} \\ + \frac{e}{m_e}\boldsymbol{E}\cdot\boldsymbol{b}\cdot\boldsymbol{p}_e^{ns} - \frac{5}{3}V_{\|e}\nabla_\| q_{\|e} = G_{\|e} - \frac{5}{2}\frac{p_e}{m_e n}F_{\|e} \end{aligned} \tag{2.68}$$

The substantial difference, however, is implicit: since we now have $\partial q_{\|e}/\partial t \sim \mathcal{O}(\delta)$, then the electron quantities $\boldsymbol{p}_e^{ns}$, $\boldsymbol{R}_e^{ns}$, $F_{\|e}$, and $G_{\|e}$ need only be calculated to first order, rather than second as for the ions.

We will occasionally find other simplifications due to the small mass-ratio, such as negligible electron gyroviscosity, but here we have presented the main points relevant for closure of our model: the small electron mass removes the necessity of evolving the electron parallel velocity as a dynamical variable, and leads to the dominance of the electron parallel heat flow.

2.4.3 Closure Requirements

By closure requirements, we mean the elements of the evolution equations that are not yet explicitly expressed in terms of the dynamical variables. They are the last remaining calculations before the system of equations is made closed and self-sufficient for analysis.

We now summarize the status of each dynamical equation. Let an equation be referred to as "complete" if all its elements are expressed in terms of only the dynamical variables. There are twelve dynamical variables in our system

$$\{n, V_{\|i}, p_i, p_e, q_{\|i}, q_{\|e}, E_\|, \boldsymbol{E}_\perp, \boldsymbol{B}\} \tag{2.69}$$

- The quasi-neutral density n is advanced through (2.20)

$$\frac{\partial n}{\partial t} + \boldsymbol{\nabla} \cdot (n\boldsymbol{V}) = 0$$

which is complete.

- The ion parallel velocity $V_{\parallel i}$ is advanced through (2.65)

$$m_i n \frac{dV_{\parallel i}}{dt} + \nabla_{\parallel} (p_i + p_e) + \boldsymbol{b} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{p}_i^{ns} + \boldsymbol{b} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{p}_e^{ns} = 0$$

which requires $\boldsymbol{p}^{ns}$ for both species to $\mathcal{O}(\delta^2)$. We will demonstrate in the course of calculation that the electron contribution to the non-scalar pressure can be neglected relative to the ion contribution on account of the small mass-ratio; thus, only the ion terms are needed to second order here.

- The two scalar pressures are advanced through an ion and an electron version of (2.60)

$$\frac{3}{2}\frac{dp_i}{dt} + \boldsymbol{\nabla} \cdot \boldsymbol{q}_i + \frac{5}{2} p_i \boldsymbol{\nabla} \cdot \boldsymbol{V}_i = 0$$
$$\frac{3}{2}\frac{dp_e}{dt} + \boldsymbol{\nabla} \cdot \boldsymbol{q}_e + \frac{5}{2} p_e \boldsymbol{\nabla} \cdot \boldsymbol{V}_e = 0$$

which are complete.

- The ion parallel heat flow is advanced through (2.63)

$$\begin{aligned}\frac{\partial q_{\parallel i}}{\partial t} &+ \frac{5}{2}\frac{p_i}{m_i}\nabla_{\parallel} T_i + \frac{5}{2}\nabla_{\parallel}\left(p_i V_i^2\right) + \boldsymbol{b} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{R}_i^{ns} - \frac{5}{2}\frac{p_i}{m_i n}\boldsymbol{b} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{p}_i^{ns} - \frac{e}{m_i}\boldsymbol{E} \cdot \boldsymbol{b} \cdot \boldsymbol{p}_i^{ns} \\ &- V_{\parallel i}\boldsymbol{V}_{\perp i} \cdot \boldsymbol{\nabla} p_i - \frac{5}{2}\boldsymbol{\nabla} \cdot \left(p_i V_{\parallel i}\boldsymbol{V}_i\right) - \frac{5}{3}V_{\parallel i}\left(\boldsymbol{\nabla} \cdot \boldsymbol{q}_i + p_i \boldsymbol{\nabla} \cdot \boldsymbol{V}_i\right) = G_{\parallel i} - \frac{5}{2}\frac{p_i}{m_i n}F_{\parallel i}\end{aligned}$$

which requires $\boldsymbol{p}_i^{ns}$, $\boldsymbol{R}_i^{ns}$, $F_{\parallel i}$, and $G_{\parallel i}$ to $\mathcal{O}(\delta^2)$.

- The electron parallel heat flow is advanced through (2.68)

$$\begin{aligned}\frac{\partial q_{\parallel e}}{\partial t} &+ \frac{5}{2}\frac{p_e}{m_e}\nabla_{\parallel} T_e + \boldsymbol{b} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{R}_e^{ns} - \frac{5}{2}\frac{p_e}{m_e n}\boldsymbol{b} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{p}_e^{ns} \\ &+ \frac{e}{m_e}\boldsymbol{E} \cdot \boldsymbol{b} \cdot \boldsymbol{p}_e^{ns} - \frac{5}{3}V_{\parallel e}\nabla_{\parallel} q_{\parallel e} = G_{\parallel e} - \frac{5}{2}\frac{p_e}{m_e n}F_{\parallel e}\end{aligned}$$

which requires $\boldsymbol{p}_e^{ns}$, $\boldsymbol{R}_e^{ns}$, $F_{\parallel e}$, and $G_{\parallel e}$ to $\mathcal{O}(\delta)$.

- The three components of the magnetic field are determined from Faraday's law (2.31)

$$\frac{\partial \boldsymbol{B}}{\partial t} = -\boldsymbol{\nabla} \times \boldsymbol{E}$$

which is complete.

- The two perpendicular components of the electric field are determined from the sum of the ion and electron versions of (2.62)

$$m_i n \frac{d\boldsymbol{V}_E}{dt} + m_i n \frac{d\boldsymbol{V}_{di}}{dt} + \boldsymbol{\nabla}_\perp (p_i + p_e) + (\boldsymbol{\nabla} \cdot \boldsymbol{p}_i^{ns} + \boldsymbol{\nabla} \cdot \boldsymbol{p}_e^{ns})_\perp - \boldsymbol{J} \times \boldsymbol{B} = 0$$

where again the non-scalar stress tensors are needed to second order (only first order for the electrons)

- The parallel electric field is given by (2.66)

$$enE_\parallel = F_{\parallel e} - \nabla_\parallel p_e - \boldsymbol{b} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{p}_e^{ns}$$

which requires $F_{\parallel e}$ and $\boldsymbol{p}_e^{ns}$ to $\mathcal{O}(\delta)$

The equations for these twelve dynamical variables also refer to the non-dynamical variables, which we now define as shorthand for particular functions of the dynamical variables

$$\begin{aligned}
\boldsymbol{V}_{\perp s} &\equiv \frac{1}{B} \boldsymbol{E} \times \boldsymbol{b} + \frac{T}{m_s \Omega_s} \boldsymbol{b} \times \boldsymbol{\nabla} \ln p_s \\
\boldsymbol{q}_{\perp s} &\equiv \frac{5}{2} \frac{p_s T_s}{m_s \Omega_s} \boldsymbol{b} \times \boldsymbol{\nabla} \ln T_s \\
\boldsymbol{J} &\equiv \boldsymbol{\nabla} \times \boldsymbol{B} \\
V_{\parallel e} &\equiv V_{\parallel i} - \frac{1}{ne} \boldsymbol{b} \cdot \boldsymbol{\nabla} \times \boldsymbol{B}
\end{aligned}$$

It is clear, then, that the set

$$\{\boldsymbol{p}^{ns}, \boldsymbol{R}^{ns}, F_\parallel, G_\parallel\} \tag{2.70}$$

is necessary and sufficient for completion of all of the above equations. The ion quantities must be calculated including $\mathcal{O}\left(\delta^2\right)$ contributions, but for the electrons, only $\mathcal{O}(\delta)$ is necessary. The remainder of this paper is dedicated to the systematic calculation of these closure requirements in a manner consistent with all adopted orderings.

Chapter 3

Other Fluid Closures

Before embarking on the main calculations, we first compare the system we have outlined with previously developed closures. We have already mentioned in Section 2.3.1 different forms for the heat flow in other closures. Here, we briefly compare approaches to stress tensors and friction, introducing a context for later calculations.

3.1 MHD

MHD is a $\mathcal{O}(1)$ system, neglecting all $\mathcal{O}(\delta)$ corrections: the velocity is dominated by $\boldsymbol{V}_E \sim v_t$, and the negligible deviations from a shifted Maxwellian

$$f^{MHD} = f_M(\boldsymbol{v} - \boldsymbol{V}) + \mathcal{O}(\delta)$$

lead to vanishing heat flow and an isotropic pressure tensor $\boldsymbol{p}^{MHD} = p\boldsymbol{I}$. Also, MHD uses equations summed over species, so that both collisional friction and energy exchange between ions and electrons are canceled, leaving no explicit reference to collisional moments. Not surprisingly, none of the closure requirements of our system, (2.70), appear in MHD, which is intended for very different plasma scales. The modifications introduced by "resistive MHD", while substantially changing the details of ideal MHD, do not alter the basic closure. In other words, adding a term $\eta\boldsymbol{J}$ doesn't change how the model is closed since $\boldsymbol{J}$ (or, depending on preference, $\boldsymbol{\nabla} \times \boldsymbol{B}$) is already a dynamical variable of MHD[14].

3.2 Braginskii Closure

The Braginskii system is a rigorous perturbative closure for a magnetized plasma based, in addition to large gyrofrequency $\Omega \gg \omega_t$ on large collisionality

$$\nu \gg \omega_t \tag{3.1}$$

It is applicable to the large flow regime

$$V \sim v_t$$

but, unlike MHD, includes finite-Larmor-radius physics [1]. These effects are obtained through the first order corrections to the distribution function due to gyration. For such a system, collisions and gyration dominate, leading to a first-order kinetic equation for $f_1 \sim \mathcal{O}(\delta)$ which can be approximately solved for f_1 in various ways [16]. Among the major consequences of this correction, in addition to purely collisional effects, are *gyration*-induced particle flow, heat flow, friction, and non-scalar pressure, known as "gyroviscosity". While the complete Braginskii system is not appropriate for weakly collisional plasmas, it gives several different contributions to $\boldsymbol{q}$, $\boldsymbol{F}$, and $\boldsymbol{p}^{ns}$, some of which are independent of the collision frequency, and collisions altogether. The implication, confirmed in experience, is that these gyration phenomena will survive relaxation of (3.1) and appear in similar form for other collisionality regimes.

The largest terms, however, originate from collisional phenomena. A representative example is the Braginskii conductive heat flux

$$\boldsymbol{q}^B = -\kappa_\parallel \boldsymbol{\nabla}_\parallel T + \kappa_\wedge \boldsymbol{b} \times \boldsymbol{\nabla} T - \kappa_\perp \boldsymbol{\nabla}_\perp T \tag{3.2}$$

with thermal conductivities (relative to $\kappa_0 \equiv nT/m$)

$$\kappa_\parallel \sim \frac{1}{\nu}\kappa_0$$
$$\kappa_\wedge \sim \frac{1}{\Omega}\kappa_0$$
$$\kappa_\perp \sim \frac{\nu}{\Omega^2}\kappa_0$$

so that, since $\Omega \gg \nu$ even for large collisionality,

$$\kappa_\parallel \gg \kappa_\wedge \gg \kappa_\perp \tag{3.3}$$

Thus, the dominant term is parallel heat conduction, which, since the flow is down the temperature gradient, is a dissipative flow that relaxes the driving gradient.

In the transition to a regime of weak collisionality, we expect new sources of heat flow to appear, not necessarily expressible as local gradients (see Section 2.3.1). The relative sizes of familiar sources, such as the temperature gradient terms above, also need not be preserved. In principle, then, using local gradients to describe heat flow in a high-temperature confinement plasma strains the validity of the model. Of course, our use of $q_\parallel$ as a dynamical variable avoids the difficult task of guessing an accurate form for the parallel heat flow in the low collisionality regime.

Relative to our model, the Braginskii system closes one order lower in the moment hierarchy, at $\partial p/\partial t$ using expressions such as (3.2) in $\boldsymbol{\nabla} \cdot \boldsymbol{q}$. However, the two systems have in common the closure requirement of the momentum evolution equation: the non-scalar pressure tensor. As was the case for the heat flow, the Braginskii result includes both collisional and gyration effects, denoted symbolically

$$\boldsymbol{p}^B = p\boldsymbol{I} + \boldsymbol{p}_\nu^B + \boldsymbol{p}_g^B$$

The collisional terms $\boldsymbol{p}_\nu^B$ are described by two collisional viscosity scales

$$\eta_{(0)} \gg \eta_{(1,2)}$$

which both depend on ν. The latter term $\boldsymbol{p}_g^B$ is known as "gyroviscosity" since it is independent of collisions, and again appears in plasmas outside the collision-dominated regime. The scale for gyroviscosity $\eta_{(3,4)}$, in the Braginskii orderings, lies in between the two conventional viscosities

$$\eta_{(0)} \gg \eta_{(3,4)} \gg \eta_{(1,2)}$$

much as the diamagnetic thermal conductivity compared with the two dissipative conductivities in (3.3). As expected, and will be shown, the collisional contributions become negligible compared to gyroviscosity in the opposite collisionality regime $\nu \ll \omega_t$.

In the Braginskii system, the structure of both the collisional and gyroviscous tensors is based on the "rate-of-strain tensor"

$$\boldsymbol{W} \equiv \boldsymbol{\nabla V} + (\boldsymbol{\nabla V})^t - \frac{2}{3}(\boldsymbol{\nabla} \cdot \boldsymbol{V})\,\boldsymbol{I}$$

where the transpose is indicated in the second term. In the applied orderings, the dominant source of the non-isotropic flow of momentum comes from the spatial variation of the large plasma flow velocity. However, for regimes where the flow velocities are not $\mathcal{O}(1)$, but rather are first order and comparable with first order heat flow, such as in the drift ordering (1.1), we expect heat flow terms to also appear as sources of viscosity. For the gyroviscosity, which dominates over collisional viscosity for confined plasmas, one finds a very similar tensor structure. While we will rigorously derive the non-scalar pressure tensor in later chapters, much of the gyroviscosity tensor can be reproduced through the replacement

$$p\boldsymbol{V} \to p\boldsymbol{V} + \frac{2}{5}\boldsymbol{q}$$

in the Braginskii form, reflecting the fact that it is variation in the energy flow

$$\boldsymbol{Q} \sim \boldsymbol{q} + \frac{5}{2} p \boldsymbol{V}$$

that is the primary source of gyroviscosity.

The friction force is also a closure requirement in common with the Braginskii system. Essentially all of the above comments again apply, most importantly the appearance of parallel heat flow as a source of collisional friction. We will more fully address the nature of the friction forces for a weakly collisional plasma in later sections, and now instead turn to other familiar closures.

3.3 FLR Truncation Closure

Perhaps the simplest model to capture guiding center dynamics in a drift ordering is the truncation given by

$$\overline{f} = f_{\parallel v} = \frac{m V_\parallel}{T} v_\parallel f_M \sim \mathcal{O}(\delta) \tag{3.4}$$

which, as discussed in 2.3.1, allows for parallel particle flow $V_\parallel$ in the model[14]. While parallel heat flow is also simple to include (using $f_{\parallel q}$ as in (2.47)), because of the odd parallel velocity symmetries, neither of these terms contributes directly to the pressure tensor integral (an even moment of f). A fluid calculation of gyroviscosity can provide higher order corrections to $\boldsymbol{p}^{ns}$ that include some effects of the parallel flows. Recall, however, that the small-δ analysis used to determine perpendicular moments is powerless to determine the homogeneous "parallel" terms [14], which turn out to be important. In other words, we will find that there are critical contributions (i.e. the largest) to the pressure tensor that have a "gyrotropic" (CGL) form

$$\left\|\boldsymbol{b} \times \boldsymbol{p}_{gt}\right\| = \boldsymbol{b} \times \boldsymbol{p}_{gt} + \left(\boldsymbol{b} \times \boldsymbol{p}_{gt}\right)^{t} = 0 \tag{3.5}$$

and as such cannot be determined through $\delta \ll 1$ manipulation of the fluid equations. The origin of this anisotropy is, of course, the intrinsic anisotropy of a plasma confined by toroidal magnetic fields. Equation (3.4), deficient in that it does not provide a means for calculating pressure anisotropy, must be supplemented, as in (2.52). We next turn to fluid closures that treat anisotropy with more care.

3.4 Neoclassical Closure

Neoclassical theory is — not surprisingly, given the status of the tokamak in the history of confined plasma research — a large and well-developed branch of plasma physics. The entirety of the next chapter is dedicated to the details plasma dynamics in toroidal fields, so here we will only briefly compare results relevant to the closure requirements of our system. Also, for brevity, we examine only the most directly comparable collisionality regime: the $\nu \ll \omega_t$ limit known as the "banana regime."

In weakly collisional magnetized plasmas, particles travel large distances along the magnetic field between collisions, but have only small (gyroradius) excursions away from field lines. In a toroidal field structure, this leads to the concept of nested flux surfaces, on which orbits are roughly confined[1]. The inhomogeneity of the field leads to magnetic trapping and the fracture of phase space into qualitatively different regions, "trapped" and "untrapped", separated by a sharp boundary layer. It is reasonable to expect that strong anisotropies can

[1]This picture is an approximate one with many qualifications, but suffices for our purposes here.

develop under these highly inhomogeneous conditions. In fact, one of the most important elements of neoclassical theory is the prediction of important pressure anisotropy. As we will see, this anisotropy plays a critical role in many neoclassical results, including both the radial particle and heat flows and also the parallel ("bootstrap") current.

First, we establish the terminology of anisotropic tensors. Consider the generalization of an isotropic pressure tensor

$$\boldsymbol{p} = p\boldsymbol{I} \rightarrow \begin{pmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{pmatrix}$$

to allow for a diagonal but anisotropic pressure

$$\boldsymbol{p}_{gt} = p_\perp \left(\boldsymbol{I} - \boldsymbol{bb}\right) + p_\| \boldsymbol{bb} \rightarrow \begin{pmatrix} p_\perp & 0 & 0 \\ 0 & p_\perp & 0 \\ 0 & 0 & p_\| \end{pmatrix}$$

where we have aligned the last coordinate with the magnetic field, and the subscript gt stands for "gyrotropic". The two directions perpendicular to the field are equivalent by symmetry and are described by $p_\perp$. The parallel component $p_\| \equiv \boldsymbol{b}\cdot\boldsymbol{p}\cdot\boldsymbol{b}$ is allowed to differ from $p_\perp$, leading to a pressure anisotropy

$$\Delta p \equiv p_\| - p_\perp \tag{3.6}$$

This form, where the two perpendicular directions of gyromotion are equivalent, but the parallel is different gives rise to the name "gyrotropic", as opposed to isotropic. It is convenient, because we have established the scalar pressure as a dynamical variable and Δp as the measure of the departure from isotropy, to rewrite the pressure in terms of

$$p \equiv \frac{1}{3} Tr\left(\boldsymbol{p}\right) = \frac{2}{3} p_\perp + \frac{1}{3} p_\|$$

and Δp, so that

$$\boldsymbol{p}_{gt} = p\boldsymbol{I} + \Delta p \left(\boldsymbol{bb} - \frac{1}{3}\boldsymbol{I}\right) \tag{3.7}$$

The relevance of this form follows from the fact that it is the general solution to the particular operation in (3.5)

$$\left\|\boldsymbol{b} \times \boldsymbol{p}_{gt}\right\| = 0$$

and therefore can be quite large, since it is not restricted by $\delta \ll 1$ arguments [14]. For later use, we take the exact divergence of $\boldsymbol{p}_{gt}$

$$\boldsymbol{\nabla} \cdot \boldsymbol{p}_{gt} = \boldsymbol{\nabla} p + \left(\boldsymbol{\nabla}_{\|} - \frac{1}{3}\boldsymbol{\nabla}\right) \Delta p + \Delta p B \nabla_{\|} \frac{\boldsymbol{b}}{B} \tag{3.8}$$

where we have made use of $\boldsymbol{\nabla}\cdot\boldsymbol{B} = \boldsymbol{\nabla}\cdot(B\boldsymbol{b}) = 0$. Similar comments apply in the construction of the gyrotropic heat tensor

$$\boldsymbol{r}_{gt} = r\boldsymbol{I} + \Delta r \left(\boldsymbol{bb} - \frac{1}{3}\boldsymbol{I}\right) \tag{3.9}$$

The non-scalar parts are obviously

$$\begin{aligned} \boldsymbol{p}_{\Delta} \equiv \boldsymbol{p}_{gt}^{ns} = \Delta p \left(\boldsymbol{bb} - \frac{1}{3}\boldsymbol{I}\right) \\ \boldsymbol{r}_{\Delta} \equiv \boldsymbol{r}_{gt}^{ns} = \Delta r \left(\boldsymbol{bb} - \frac{1}{3}\boldsymbol{I}\right) \end{aligned} \tag{3.10}$$

Neoclassical theory predicts important anisotropies, the essential result for the banana regime taking the form [16, 20]

$$\begin{aligned} \left\langle \boldsymbol{B} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{p}_{\Delta} \right\rangle = \alpha_1 V_\theta + \alpha_2 \frac{2q_\theta}{5p} \\ \left\langle \boldsymbol{B} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{r}_{\Delta} \right\rangle = \alpha_2 V_\theta + \alpha_3 \frac{2q_\theta}{5p} \end{aligned} \tag{3.11}$$

where we have introduced the poloidal components V_θ and q_θ, and subsumed the remaining details (which are many) into the coefficients α_i. Two comments are appropriate. The first is that here the *flux-surface-averaged* anisotropies are specified in terms of the poloidal flows, thereby obtaining closure for the *flux-surface-averaged* fluid equations. The second is that, up to geometrical factors, the anisotropy is formally first order in δ, e.g.

$$\frac{1}{B}\langle \boldsymbol{B} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{p}_{\Delta} \rangle \sim mn\nu\delta V_t$$

and is therefore relatively large. There is an additional factor of the collision frequency ν, which is also small; but in any case, according to Section 2.4.3, terms of this size must be included.

Neoclassical closure differs from our approach in several major aspects. It is a rigorous closure, supplementing the fluid equations with a perturbative solution of the drift-kinetic equation. However, the theory is designed for the transport regime, and as we have eluded to, requires the flux-surface averaging of the equations, which eliminates the possibility of describing phenomena varying within flux surfaces. While our model is instead appropriate for a local description of a drift ordered plasma, the results of neoclassical theory are strong evidence that a theory intended to apply to toroidal plasmas must take seriously the anisotropies arising from the complications of guiding center motions. In the next chapter we will study this subject in detail.

3.5 Neoclassical MHD

The final alternative closures we would like to mention are "neoclassical MHD" [2] and "extended MHD" [29], both of which are concerned with augmenting the appealingly simple equations of ideal/resistive MHD with neoclassical and/or finite-Larmor-radius corrections to construct descriptions more appropriate for high-temperature confined plasmas.

Neoclassical MHD is the straightforward inclusion of neoclassical effects by removal of the flux-surface average brackets. The basic recipe is outlined as follows. From (3.8) we have

$$\boldsymbol{B} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{p}_{gt} = B\nabla_{\|} p + \frac{2}{3} B\nabla_{\|}\Delta p - \Delta p \nabla_{\|} B$$

The flux-surface average, by construction, annihilates the $B\nabla_{\|}$ operator, so that

$$\left\langle \boldsymbol{B} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{p}_{gt} \right\rangle = \left\langle \boldsymbol{B} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{p}_{\Delta} \right\rangle = -\left\langle \Delta p \nabla_{\|} B \right\rangle$$

These equations are exact. Reference [2] then makes use of a closure (recall (3.11) and the fact that the heat flow is negligible in the MHD ordering)

$$\left\langle \boldsymbol{B} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{p}_{\Delta} \right\rangle = mn\mu < B^2 > \frac{V_\theta}{B_\theta} \tag{3.12}$$

so that

$$-\left\langle \Delta p \nabla_{\|} B \right\rangle = mn\mu < B^2 > \frac{V_\theta}{B_\theta}$$

This last equation is another reminder that neoclassical theory only determines averages, in this case, the product of Δp with another function, namely $\nabla_{\|} B$. It is, however, natural to experiment with less-than-strict adherence to the the flux surface average brackets. In that case, one might separate out and "solve" for the anisotropy

$$\Delta p = -\frac{mn\mu < B^2 >}{< \nabla_{\|} B >} \frac{V_\theta}{B_\theta}$$

to use it in local expressions, such as (3.8). This closure of neoclassical MHD introduces the important neoclassical corrections to standard MHD, e.g. the bootstrap current[2].

Although the above pressure anisotropy closure was derived for a transport ordered plasma, neoclassical MHD is still intended to describe an MHD ordered plasma; these orderings are different both from each other and the drift ordering chosen for our model. In addition, this treatment of flux-surface averages is perhaps cavalier; accordingly, several studies have been dedicated to constructing local expressions for $\boldsymbol{b} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{p}_{\Delta}$. These, to an extent, validate the above treatment but, however, either resort to a neoclassical/transport ordering [33] or do not present a consistent derivation from first principles [8].

3.6 Extended MHD

The use of fluid models for computation has recently been reviewed, with extensive discussion of the form of the non-scalar pressure in the various orderings (Hall MHD, MHD, drift, transport) and the implications for computational analysis[29]. "Extended MHD" is the name given there to the system(s) that add both neoclassical and gyroviscous effects (through $\boldsymbol{p}^{ns}$) to the drift ordered continuity equation $\partial \boldsymbol{n}/\partial t$, equation of motion $\partial \boldsymbol{V}/\partial t$, Ohm's law determining $\boldsymbol{E}$, and, for simplicity, use an external equation of state to specify the scalar pressure p. The neoclassical closure employed is the same as in (3.12) and [8]; in other words, intended for a MHD-like description, the viscosity does not contain the heat flow as it does in (3.11). The formal presence of the heat flow (constructed through temperature gradients) in the gyroviscous pressure tensor is, however, noted.

Reference [29] shows extended MHD is an effective model of plasma behavior; however, no unified derivation systematically determines all proper closure requirements within a single ordering choice. This is the essential purpose and result of the present work.

3.7 The Road to Closure

In summary, we have converted the exact moment equations to a nearly closed system of equations for a small number of dynamical variables (2.69) appropriate for a drift ordered plasma. The final remaining steps are the calculations of the closure requirements

(2.70. The above comparisons with other closures provided several manifestations of these that will appear in similar form in our system. However, here they will all follow systematically from (2.52). In addition, elements heretofore absent in fluid approaches, such as a dynamic and consistent parallel heat flow, will play important roles.

Chapter 4

Plasma in a Torus

This chapter develops the 'toroidal' part of the thesis title. With neoclassical theory as a contextual background, we demonstrate some aspects of plasma evolution in inhomogeneous magnetic fields with toroidal structure. Since we seek a description for plasma conditions outside the applicability of the established theory, we construct the necessary generalizations — abandoning both flux-surface averaging and use of the transport ordering — and study the main consequences for closure of the fluid system developed in Chapter 2.

As we have alluded to in Section 3.4, the appearance of anisotropy in the stress tensors[1], $\boldsymbol{P}$ and $\boldsymbol{R}$, is of primary significance. With a focus on the systematic adherence to the theoretical framework outlined so far, we now undertake a study of these anisotropies, culminating in the practical requirements of this chapter: the parallel divergences of the non-scalar stresses, $\boldsymbol{b} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{P}^{ns}$ and $\boldsymbol{b} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{R}^{ns}$.

4.1 Beyond Transport Theory

4.1.1 The Significance of Neoclassical Theory

A critical development in the history of plasma confinement was the understanding that inhomogeneities in the magnetic field had a major effect on plasma behavior, especially

[1]For simplicity, we will refer to both $\boldsymbol{P}$ and $\boldsymbol{R}$ as "stress" tensors. Also, recall from (2.11) that the difference between $\boldsymbol{P}$ and $\boldsymbol{p}$ is $\mathcal{O}(\delta^2)$, so we may interchange them when convenient in places where the difference is consistently negligible

with regard to the motion of guiding centers. An inhomogeneous field leads to guiding center drifts that do not appear in a homogeneous system; when these are taken into account, as must be done for a tokamak, the resulting transport (neoclassical) is significantly larger than that for a similar homogeneous system (classical) [16].

The drift-kinetic equation dictates the motion of guiding centers. In attacking this equation, the development of neoclassical theory required the use of substantial mathematical mechanics, but eventually provided a relatively straightforward rigorous kinetic treatment. Two ingredients in the foundation of the theory, however, make it inappropriate for our current investigation. Firstly, it was necessary to make use of flux-surface averages; contemporary investigations, however, e.g. magnetic island studies, would benefit from a local description. Secondly, neoclassical theory only applies in the transport regime, and is therefore too slow to model mild instabilities. Relaxing these restrictions allows a more generally applicable theory, but the complexity of the problem is obviously increased.

4.1.2 Magnetic Trapping is Fast Dynamics

The physical source of anisotropy is magnetic trapping: the highly anisotropic trapped/untrapped phase-space structure leads inevitably to anisotropy in a fluid description. Neoclassical theory is a detailed study of the effects of magnetic trapping, but to go beyond the restrictive transport ordering used by neoclassical theory we must take advantage of an analogy between gyro-motion and guiding center motion. The point is that the dynamics of magnetic trapping — the passing or bouncing of guiding centers — is exceedingly rapid, faster than everything except gyration[2].

[2] The definition of a magnetized plasma includes the condition that the gyrofrequency Ω be larger than all other frequency scales.

Recall the approximate picture when gyration is the fastest time-scale: particles generally complete their gyro-orbits before anything else can happen and are basically described by the motion of their gyroaveraged guiding centers with small corrections to account for the other dynamics in the presence of gyration. A similar separation of time-scales is relevant in the context of passing and trapped/bouncing particles. The measure of particle speed along field lines is essentially the thermal speed, so passing particles circle the torus at the transit frequency, ω_t. The bounce frequency of trapped particles is perhaps more subtle — they obviously slow to a halt before bouncing, and barely trapped particles have divergent bounce times. Rapid streaming is therefore mitigated by the strength of the trapping, but bouncing is still very fast dynamics. For example, a small-aspect ratio estimation for the bounce frequency is [14, 16]

$$\omega_b \sim \sqrt{\epsilon}\omega_t \tag{4.1}$$

which is essentially the transit frequency[3]. So, although the drift ordering promotes time evolution to be faster than the transport ordering, the circulation and bouncing of guiding centers remains fast

$$\omega_b \gg \frac{\partial}{\partial t} \sim \omega_\star \sim \nu \sim \delta\omega_t \tag{4.2}$$

thus indicating that guiding center motion can be treated in a perturbative manner in analogy with gyration.

Averaging out the ultra-fast gyration around magnetic field lines leaves a distribution that does not vary over on the small space and time scales of the gyroradius and

[3]To unburden the theory, ϵ is often taken to be an asymptotic small parameter, but in practice we still expect bouncing to be faster than anything else except gyration.

frequency. In other words

$$f(\boldsymbol{x}, \boldsymbol{v}) = F\left(\boldsymbol{X}, v_\parallel, v_\perp\right)$$

for some function F, where the dependence is on the slowly varying guiding center position $\boldsymbol{X} = \boldsymbol{x} - \boldsymbol{\rho}$ and there is also no dependence on gyroangle. Since the deviation $\boldsymbol{\rho}$ from the guiding center is small, it makes sense to expand

$$f(\boldsymbol{x}, \boldsymbol{v}) \approx F(\boldsymbol{x}, v_\parallel, v_\perp) - \boldsymbol{\rho} \cdot \boldsymbol{\nabla} F\left(\boldsymbol{x}, v_\parallel, v_\perp\right)$$

resulting in a heuristic version of (2.48). Thus, we are led to the implication

$$\left.\begin{aligned} \Omega &\gg \omega \\ \rho &\ll L_\perp \end{aligned}\right\} \Longrightarrow \text{gyro-averaged distribution} \tag{4.3}$$

with the first order correction

$$-\boldsymbol{\rho} \cdot \boldsymbol{\nabla} f_0 \tag{4.4}$$

where f_0 is the lowest order distribution, the $\delta \to 0$ limit.

The same reasoning applies to the description of guiding centers. Since the timescale for fast streaming and bouncing is $\omega_b \sim \omega_t$, guiding center evolution is affected primarily by the inhomogeneities of the field sampled on that fast timescale, the other dynamics of the plasma occurring at a slower rate. In other words, an orbit-averaged distribution function is still appropriate in the drift ordering. As above, the disparity in space and time scales between bouncing/passing and the bulk evolution of the plasma is large, with the implication

$$\left.\begin{aligned} \omega_b &\gg \omega \\ \Delta_b &\ll L_\perp \end{aligned}\right\} \Longrightarrow \text{orbit-averaged distribution} \tag{4.5}$$

and first-order correction

$$-\Delta_b \frac{df_0}{dr} \tag{4.6}$$

where Δ_b is the (small) width of the banana orbits and r is the radial coordinate. The banana regime correction to the distribution function, familiar from neoclassical theory, has precisely these features[16]. We will later discuss the detailed form of the correction, but the analogy in (4.3)-(4.6) is evident.

We are therefore led to the ansatz for the guiding center distribution used in (2.52)

$$\overline{f} = f_M + f_b + f_\parallel + \mathcal{O}(\delta^2) \tag{4.7}$$

where f_b is the first order banana regime solution to the drift-kinetic equation. Of course, taking just those two terms reproduces neoclassical theory — after applying the transport ordering and flux-surface averaging — so we must include $f_\parallel$ to go beyond neoclassical theory.

In summary, the rapid nature of gyration and guiding center passing/trapping dictate the basic motions of a drift-ordered plasma. Our choice for distribution function incorporates these basic processes explicitly and leaves room for the remaining slower processes to evolve on top of the background dynamics of gyro- and banana orbits.

Next we turn to a detailed discussion of the drift-kinetic equation to establish these issues in more detail.

4.2 Ordering the Drift-Kinetic Equation

Consider the full drift-kinetic equation [14]

$$\frac{\partial \overline{f}}{\partial t} + u\nabla_\parallel \overline{f} + \boldsymbol{v}_d \cdot \boldsymbol{\nabla} \overline{f} + \frac{dU}{dt}\frac{\partial \overline{f}}{\partial U} + \frac{d\mu}{dt}\frac{\partial \overline{f}}{\partial \mu} - \mathcal{C}(\overline{f}) = 0. \tag{4.8}$$

where we denote the parallel velocity $u \equiv \boldsymbol{b} \cdot \boldsymbol{v}$. An expansion in gyro-parameter,

$$\overline{f} = f_0 + f_1 + \mathcal{O}(\delta^2)$$

using the drift ordering

$$\partial/\partial t \sim v_d \sim f_1 \sim \delta$$

results in the two lowest order equations

$$u\nabla_\parallel f_0 - \mathcal{C}\left(f_0\right) = 0 \tag{4.9}$$

and

$$\frac{\partial f_0}{\partial t} + u\nabla_\parallel f_1 + \boldsymbol{v}_d \cdot \boldsymbol{\nabla} f_0 + \frac{dU}{dt}\frac{\partial f_0}{\partial U} + \frac{d\mu}{dt}\frac{\partial f_0}{\partial \mu} - \mathcal{C}\left(f_1\right) = 0 \tag{4.10}$$

where collisions have not yet been ordered. In a standard transport ordering, time variation is completely neglected compared to collisions, even if collisions are small, as in (2.55). In this case, (4.9) can be used as it stands and provides well-known solutions for f_0: Maxwellians constant over flux surfaces. Clearly the physical processes of parallel streaming and collisions determine the solutions, but the Maxwellian result does not depend on collision frequency and therefore obtains in any collisionality regime.

Our ordering (4.2) neglects neither collisions nor time variation with respect to each other, but prevents us from deriving such simple conclusions concerning f_0, since keeping $\partial f_0/\partial t \sim \mathcal{C}\left(f_0\right)$ would invalidate the arguments for flux surface Maxwellians. Physically, it is evident that there must not be any rapid time-dependent phenomena (no $\partial/\partial t \gtrsim \nu$) and that a collision time must elapse before the plasma will undergo any substantial relaxation to a Maxwellian. Therefore, since we order $\partial/\partial t \sim \nu$ our model cannot capture lowest-order Maxwellianization. However, our fluid model is not intended to apply to far-from-equilibrium conditions, but instead to the relatively slow evolution of a plasma in an already established near-equilibrium state. Such conditions are observed to develop in tokamak plasmas. Even under the onset of turbulence or mild instabilities the plasma can be usefully described as Maxwellian with small, but crucial, corrections (such as gyration and bounce motion). Thus, we do not attempt to rigorously derive the lowest order distribution function, but instead use the common *ansatz* that f_0 is Maxwellian. Since plasma experiments often confirm and fluid theories generally rely on such assumptions, we believe it to be a reasonable element to include in our theory.

Equation (4.9) is then satisfied identically, and separating the f_1 and $f_0 = f_M$ terms in the first-order equation (4.10) gives

$$u\nabla_{\|} f_1 - \mathcal{C}(f_1) = -\frac{\partial f_M}{\partial t} - \frac{dU}{dt}\frac{\partial f_M}{\partial U} - \boldsymbol{v}_d \cdot \boldsymbol{\nabla} f_M. \tag{4.11}$$

If the right-hand side were to vanish, this equation would reduce to (4.9) and give the same Maxwellian solutions. Thus, the derivatives of f_M can be considered the source terms which drive the solutions for f_1 away from Maxwellians. As mentioned, transport analysis of (4.11) in the banana regime neglects the time derivatives, resulting in a kinetic equation [14, 16]

$$u\nabla_{\|} f_b - \mathcal{C}(f_b) = -\boldsymbol{v}_d \cdot \boldsymbol{\nabla} f_M - \frac{\mathcal{Z}e}{T} uE_{\|} f_M \tag{4.12}$$

predicting familiar banana motion with the widths of the banana orbits originating from the $\boldsymbol{v}_d \cdot \boldsymbol{\nabla} f_M$ term[4]. Informally, this reduction occurs as a result of time-scale separation: the bulk motion of the plasma is neglected on the scales of gyration and bounce motion. Since we seek a description that focuses on gyration and bounce motion, this effect is of primary importance.

To explicitly extract the banana regime response from the complete first order distribution function we write

$$f_1 = f_b + f_r$$

[4]Technically, the $E_{\|}$ term appears in the electron version of this equation, but not in the ion version.

where f_b is governed by (4.12) and the equation for the remainder, f_r, evidently includes everything else, the difference in the right-hand sides of (4.12) and (4.11)

$$u\nabla_{\|} f_r - \mathcal{C}(f_r) = -\frac{\partial f_M}{\partial t} - \frac{dU}{dt}\frac{\partial f_M}{\partial U} + \frac{\mathcal{Z}e}{T} u E_{\|} f_M$$

The familiar solutions of (4.12) are presented in the following sections, but we do not attempt to solve the above kinetic equation for f_r. Instead, as it represents whatever is occurring other than gyration and bouncing, f_r is subsumed into the parts of the distribution function that must be represented by the evolution of the fluid moments. In other words, we model the solution through the use of our truncation closure. By this separation, the lowest order bounce dynamics appear directly and distinctly through f_b, which is only part of the complete distribution; the more general flows with sources outside gyration and bouncing are captured using $f_{\|}$.

4.2.1 Neoclassical Time Dependence

Another note regarding time variation is worth mentioning here. In the literature there are numerous investigations of time dependence in a neoclassical context. One approach is to refrain from placing any ordering restrictions on the time derivative [unlike our use of (4.2) or (2.55)] and to examine the time dependence of the solutions to the resulting differential equations. Such studies of the relaxation of poloidal flow have resulted in a variety of time scales characterizing $\partial/\partial t$ [7, 19, 21, 24]. Results usually fall somewhere in the interval

$$\frac{\nu}{\epsilon} \gtrsim \frac{\partial}{\partial t} \gtrsim \nu.$$

Under our adopted frequency ordering,

$$\frac{\nu}{\epsilon} \gg \frac{\partial}{\partial t} \sim \omega_\star \sim \nu \sim \delta\omega_t. \tag{4.13}$$

The condition of large aspect ratio, $\epsilon \ll 1$, allows a large difference between ν and ν/ϵ, providing a major simplification in our treatment of the banana kinetic equation. In particular,

$$\frac{\partial f_b}{\partial t} \sim \omega_\star f_b$$

is small compared to

$$\mathcal{C}\left(f_b\right) \sim \frac{\nu}{\epsilon} f_b$$

and $\partial f_b/\partial t$ does not enter the equation for the banana distribution (4.12). Notice that while the proper frequency scale for $\mathcal{C}\left(f_b\right)$ is ν/ϵ, other collisions are ordered by

$$\mathcal{C}\left(f_g\right) \sim \nu f_g$$

$$\mathcal{C}\left(f_\parallel\right) \sim \nu f_\parallel .$$

Thus the particular sensitivity of banana orbits to collisions makes time dependence of secondary importance for determining f_b. The same is not true for other parts of the distribution function, however.

In summary, (4.2) prevents our model from capturing effects that proceed as fast as ν/ϵ, but greatly simplifies the treatment of neoclassical time-dependence and still allows for time variation at the drift frequency, $\omega_\star$, and treatment of a number of interesting phenomena such as ITG modes and slow evolution of unstable magnetic islands.

Next we examine in more detail the functional form of the solutions to (4.12), the banana regime distribution function, f_b.

4.3 Banana Regime Distribution Function

There are many references treating the banana regime distribution function, e.g. [14, 16, 28], so we will only take a minimal tour, focusing on aspects necessary for the anisotropy calculation. A short description of toroidal coordinates, however, conveniently introduces many notations and quantities for later use.

4.3.1 Toroidal and Pitch-Angle Coordinates

We use toroidal coordinates (r, θ, ζ) where the radius r labels magnetic surfaces, and θ and ζ are the poloidal and toroidal angles respectively. For example, the *equilibrium* magnetic field in an *axisymmetric* toroidal system has the form

$$\boldsymbol{B} = I(r)\boldsymbol{\nabla}\zeta + \boldsymbol{\nabla}\zeta \times \boldsymbol{\nabla}\chi(r) \tag{4.14}$$

where χ measures the poloidal flux. The major radius is denoted by R; its maximum value is R_0. Note that $|\boldsymbol{\nabla}\zeta| = 1/R$, so that the toroidal magnetic field has magnitude

$$B_T = \frac{I}{R}.$$

The total (equilibrium) field magnitude is expressed as

$$B(r, \theta) = \frac{B_0}{h(r, \theta)} \tag{4.15}$$

where B_0 is a constant. We will often use a large-aspect ratio approximation in which

$$\epsilon \equiv \frac{r}{R_0}$$

is assumed small. Then $B \approx B_T$ whence

$$h \approx 1 + \epsilon \cos\theta. \tag{4.16}$$

The frequency that appears most commonly in neoclassical theory is the poloidal gyrofrequency, which is conveniently defined by

$$\Omega_p \equiv \left(\frac{\chi'}{I}\right)\frac{eB_0}{mc}. \tag{4.17}$$

Here the prime denotes a radial derivative. Note that (4.14) is only valid in the special case of axisymmetric equilibrium; we use this form only to discuss canonical neoclassical results, and not for our general analysis.

Convenient velocity variables are the gyrophase angle, γ, and the approximate invariants $w = v^2/2$ and λ, the pitch-angle variable:

$$\lambda \equiv \frac{B_0 v_\perp^2}{B v^2} = h \frac{v_\perp^2}{v^2}.$$

Thus we have

$$\boldsymbol{v} = \boldsymbol{b} u(\lambda, w) + v_\perp(\lambda, w) \left(\boldsymbol{e}_1 \cos\gamma - \boldsymbol{e}_2 \sin\gamma \right) \tag{4.18}$$

where $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{b})$ forms an orthogonal triplet of unit vectors, and denoting the sign of the parallel velocity by σ, the parallel velocity can be written

$$u = \sigma \sqrt{2w} \sqrt{1 - \frac{\lambda}{h}}.$$

The maximum value of λ is evidently

$$\lambda_{\max} = h.$$

The smallest λ that allows $u = 0$ on a given flux surface marks the boundary between the trapped and passing regions of velocity space. Denoting this value by λ_c we have

$\lambda < \lambda_c$: untrapped or passing region;

$\lambda \geq \lambda_c$: trapped region.

In the large-aspect ratio case, $\lambda_c = 1 - \epsilon$.

Additionally, we will require the Jacobian for pitch-angle coordinates [14],

$$\int d^3 v = \frac{2\pi}{h} \sum_\sigma \int_0^h d\lambda \int_0^\infty \frac{w dw}{|u|} \tag{4.19}$$

4.3.2 Trapping Function

We next introduce the function $u^*(r, \theta, \lambda, w)$ such that

$$h\, u^*(r, \theta, \lambda, w) \equiv \sigma \sqrt{2w}\, L(r, \theta, \lambda)$$

where

$$L(\theta, \lambda) \equiv h\sqrt{1 - \frac{\lambda}{h}} - \frac{\Theta(\lambda_c - \lambda)}{2} \int_{\lambda}^{\lambda_c} \frac{d\lambda'}{\langle \sqrt{1 - \frac{\lambda'}{h}} \rangle}. \tag{4.20}$$

Here Θ denotes a step-function and the angular brackets denote a flux-surface average, which has the form

$$\langle \mathcal{F} \rangle \equiv \oint d\theta \mathcal{F} h^2 \Big/ \oint d\theta h^2. \tag{4.21}$$

It is important to notice that the function L is localized in λ, being roughly concentrated in the trapped particle region. This follows from the observation that, as $\lambda \to 0$, the flux-surface average in (4.20) has no effect; then the λ-integral in L can be performed and the two terms in (4.20) can be seen to cancel:

$$\lim_{\lambda \to 0} L(\lambda) = 0.$$

On the other hand the localization is weak: L has a tail, ordered by $\sqrt{\epsilon}$, that extends into the passing region.

It follows that the function u^* is a sort of truncated version of the parallel velocity u. Both u and u^* have the same energy and σ dependence, and they precisely coincide in the trapped region (where the step-function vanishes). They differ only because u^*, being localized to the trapped region, decays away for small λ, while u increases as $\lambda \to 0$.

4.3.3 Distribution Function

The final ingredients to define are traditional driving forces for species s

$$\begin{aligned} A_{1s} &= \frac{\partial \ln p_s}{\partial r} + \frac{\mathcal{Z}_s e}{T_s}\frac{\partial \Phi}{\partial r} \\ A_{2s} &= \frac{\partial \ln T_s}{\partial r} \end{aligned} \tag{4.22}$$

and the neoclassical flow parameter

$$\alpha \approx 1.32705 \approx \frac{4}{3}.$$

We can then explicitly write approximate solutions — valid through first order in the poloidal gyroradius, Ω_p, and zeroth order in the collision frequency — to the banana kinetic equation (4.12). We take the distributions from Reference [28] and note that, for both ions and electrons, the solutions contain a part that is weakly localized to the trapped region of λ-space, denoted f^{loc} and a non-localized part f^{non}

$$f_b = f^{non} + f^{loc}. \tag{4.23}$$

The banana regime neoclassical ion distribution is given by

$$\frac{f_{bi}}{f_{Mi}} = -\frac{hu}{\Omega_{pi}}\left[A_{1i} + \left(\alpha - \frac{5}{2}\right)A_{2i}\right] - \frac{hu^*}{\Omega_{pi}}\left(\frac{m_i w}{T_i} - \alpha\right)A_{2i}. \tag{4.24}$$

The first term in brackets, corresponding to a Maxwellian displaced in parallel velocity, is the non-localized part

$$f_i^{non} \equiv -f_{Mi}\frac{hu}{\Omega_{pi}}\left[A_{1i} + \left(\alpha - \frac{5}{2}\right)A_{2i}\right].$$

The remaining term is localized to the trapped region because of u^*

$$f_i^{loc} \equiv -f_{Mi}\frac{hu^*}{\Omega_{pi}}\left(\frac{m_i w}{T_i} - \alpha\right)A_{2i}. \tag{4.25}$$

The electron distribution is more complex, due to the electron response to the parallel electric field and collisions with ions, but we will not need the non-localized part, so we only write

$$f_e^{loc} = -f_{Me}\frac{hu^*}{\Omega_{pe}}\left[\left(1+\frac{T_i}{T_e}\right)\frac{n'}{n} - \left(\frac{3}{2}-\frac{m_e w}{T_e}\right)\frac{T_e'}{T_e} - 0.172\frac{T_i'}{T_e}\right] + hu^* E_\parallel f_S \tag{4.26}$$

where the function, f_S, is the solution to the Spitzer problem[5]

$$\mathcal{C}(uf_S) = -\frac{e}{T}uf_M \tag{4.27}$$

4.3.4 Banana Flows and Parallel Flow Compensation

To demonstrate the basic elements of calculation using the above neoclassical distribution functions, we now determine the contributions to the *ion* parallel flows, namely[6]

$$\begin{aligned} V_{\parallel bi} &\equiv V_\parallel[f_{bi}] = \frac{1}{n}\int d^3v\; uf_{bi} \\ q_{\parallel bi} &\equiv q_\parallel[f_{bi}] = \frac{p_i}{n}\int d^3v\; \left(\frac{m_i w}{T_i} - \frac{5}{2}\right) uf_{bi} \end{aligned} \tag{4.28}$$

The contribution from the non-localized distribution only involves elementary integrals, such as

$$\int_0^\infty dx\, x^n e^{-x^2} = \frac{1}{2}\Gamma\left(\frac{n+1}{2}\right) = \begin{cases} \left(\frac{n-1}{2}\right)!/2 & n \text{ odd} \\ \frac{1}{2}\sqrt{\pi}\,(n-1)!!/\sqrt{2^n} & n \text{ even} \end{cases} \tag{4.29}$$

for non-negative integers n. The contribution from the localized distribution, however, requires integration of the trapping function, which can be approximated

$$\int_0^h d\lambda\, L = \frac{2}{3}\left(\sqrt{2\epsilon} + h^2 - 1\right) + \mathcal{O}(\epsilon) \tag{4.30}$$

[5]We will discuss f_S in more detail later when it will be necessary for computation.
[6]This expression for $q_\parallel$ follows from (2.59).

Combining these results gives the neoclassical ion parallel particle and heat fluxes

$$\begin{aligned} V_{\parallel bi} &= -\frac{hT_i}{m_i\Omega_{pi}}\left[A_{1i} + \left(\alpha - \frac{5}{2}\right)\left(\frac{1-\sqrt{2\epsilon}}{h^2}\right)A_{2i}\right] \\ q_{\parallel bi} &= -\frac{5}{2}p_i\frac{hT_i}{m_i\Omega_{pi}}\left(\frac{\sqrt{2\epsilon}+h^2-1}{h^2}\right)A_{2i} \end{aligned} \tag{4.31}$$

Similar results apply to for the electrons. These flows, which occur because of banana motion, are analogous to the perpendicular flows induced because of gyration; they are both generic, in the sense that they occur whatever dynamical process may be underway, as long as the orderings (4.3) and (4.5) are satisfied. They are not the complete flows, however.

We have demonstrated that f_b explicitly contributes to the parallel flows, but our truncation closure requires that the distribution function be parametrized in terms of the total flows. We must therefore modify the form of $f_\parallel$, subtracting off the neoclassical part of the flow provided by f_b, so that only the total flow is returned when integrating the total distribution function. This is accomplished with the substitution

$$V_\parallel \to V_\parallel - V_{\parallel b}$$

$$q_\parallel \to q_\parallel - q_{\parallel b}$$

into (2.47) with the result

$$\begin{aligned} f_{\parallel v} &\equiv -\frac{m}{T}\left(V_\parallel - V_{\parallel b}\right)uf_M \\ f_{\parallel q} &\equiv -\frac{2m}{5pT}\left(\frac{mv^2}{2T} - \frac{5}{2}\right)\left(q_\parallel - q_{\parallel b}\right)uf_M \end{aligned} \tag{4.32}$$

which ensures

$$\int d^3v\ v_\parallel\left(f_\parallel + f_b\right) = V_\parallel$$

$$\int d^3v\ \frac{1}{2}mv_r^2 v_{r\parallel}\left(f_\parallel + f_b\right) = q_\parallel$$

4.4 Anisotropy

With the distribution function for the ions and electrons now specified, we can begin the calculation of the stress anisotropies due to guiding center motion in toroidal fields. Because of the parallel velocity symmetry shared by u and u^*, f_b does not contribute to the density

$$n[f_b] = 0$$

as required by our use of the Maxwellian, f_M, to convey the total density. However, by the same symmetries, all components of stress tensor vanish

$$\boldsymbol{P}[f_b] = \int d^3v \; m\boldsymbol{v}\boldsymbol{v} f_b = 0 \tag{4.33}$$

because the products $\boldsymbol{v}\boldsymbol{v}u$ and $\boldsymbol{v}\boldsymbol{v}u^*$ are always odd in at least one velocity component; all components of $\boldsymbol{R}[f_b]$ also vanish. Clearly, there is no anisotropy conveyed through the distribution functions we have written down. This null result is perhaps surprising — given the purpose of this chapter — until we recall that the solutions presented in (4.24) and (4.26) were only $\mathcal{O}(\nu^0)$ approximations to the exact solutions; the source of anisotropy must therefore be higher order in ν and we require higher order corrections of the true solution.

4.4.1 Collisional Correction

We have seen that the $\mathcal{O}(\nu^0)$ neoclassical distribution functions do not contribute to the tensors $\boldsymbol{P}$ and $\boldsymbol{R}$; that the lowest non-vanishing order is $\mathcal{O}(\nu^1)$ follows from the kinetic equation. Expanding in collision frequency, $f = f^0 + f^1 + f^2 + \ldots$, the first order version of (4.12) becomes

$$u\nabla_{\|} f^1 = \mathcal{C}(f_b)\,. \tag{4.34}$$

Without explicitly solving (4.34), we note that its solution will be even in u because f_b is odd. The contributions to the stress tensors, therefore, will not vanish

$$\boldsymbol{P}[f^1] \neq 0$$

$$\boldsymbol{R}[f^1] \neq 0$$

and, as we will see, stress anisotropies will enter, proportional to ν.

In light of the effort required to solve for the $\mathcal{O}(\nu^0)$ part of the distribution, solving (4.34) is presumably quite a difficult task. We instead take advantage of the fact that the equation that defines the full, exact solution, (4.12), includes information to all orders, not just the lowest order approximations we have been discussing. This is accomplished by noticing that the parallel divergences of the stress tensors can be written in terms of parts of the kinetic equation itself. Consider, for example, the gyrotropic pressure tensor expressed in terms of $p_\parallel$ and $p_\perp$

$$\boldsymbol{p}_{gt} = p_\perp (\boldsymbol{I} - \boldsymbol{bb}) + p_\parallel \boldsymbol{bb}$$

This expression has the exact parallel divergence

$$\boldsymbol{b} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{p}_{gt} = \nabla_\parallel p_\parallel - (p_\parallel - p_\perp)\nabla_\parallel \ln B \tag{4.35}$$

Applying the product rule

$$\nabla_\parallel (uf) = u\nabla_\parallel f - (mu)^{-1} f\mu\nabla_\parallel B$$

where the parallel gradient is taken at constant energy and μ, and then integrating over velocity space gives

$$\int d^3v mu^2 \nabla_\parallel f = \int d^3v mu \nabla_\parallel (uf) + \int d^3v f\mu\nabla_\parallel B.$$

Recalling the definition of $\mu \equiv mv_\perp^2/2B$ and $p_\perp$,

$$p_\perp = \int d^3v f \mu B,$$

we see that

$$\int d^3v mu^2 \nabla_\| f = \frac{1}{h}\nabla_\|(hp_\|) + p_\perp \nabla_\| \ln B$$

Demonstrating the first term on the right-hand side is the only non-triviality, requiring only a little manipulation of the integral and the pitch-angle Jacobian, (4.19). In view of (4.35), we finally have for the parallel divergence of the stress tensor[7]

$$\boldsymbol{b} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{P}_{gt} = \int d^3v\ mu^2 \nabla_\| f. \tag{4.36}$$

This equation is, of course, not merely pulling the derivative inside the integral for $\boldsymbol{P}_{gt}$. The derivative on the left-hand side is on a purely spatial function, while the parallel gradient on the right-hand side is at constant μ and total energy, precisely the operator in the kinetic equation.

Inserting the banana distribution kinetic equation, (4.12), gives the contribution due to the guiding center motion

$$\boldsymbol{b} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{P}_b = \int d^3v\ mu \left[C(f_b) - \frac{\mathcal{Z}e}{T} u E_\| f_M \right] \tag{4.37}$$

where the $\boldsymbol{v}_D \cdot \boldsymbol{\nabla} f_M$ term drops out because of antisymmetry in u, and the subscript b indicates either "bounce" or "banana". The same reasoning can be applied to the energy-weighted stress tensor, which however, contains an extra factor of w

$$\begin{aligned} \boldsymbol{b} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{R}_b &= \int d^3v\ mwu^2 \nabla_\| f_b \\ &= \int d^3v\ muw \left[C(f_b) - \frac{\mathcal{Z}e}{T} u E_\| f_M \right] \end{aligned} \tag{4.38}$$

[7] According to (2.11) we have in several places, dropped the small distinction between $\boldsymbol{P}$ and $\boldsymbol{p}$ as higher order in δ.

as expected from the definition. Notice that it is only necessary to use the $\mathcal{O}(\nu^0)$ part of f_b because the action of the collision operator introduces a factor of ν and makes the result first order. In this manner, the effect of first order corrections are captured without an explicit first order solution.

An important feature of these results is that there is no flux-surface averaging required, as in neoclassical theory; all expressions are local and their validity is not restricted to the transport ordering.

We now perform the computations on the right-hand sides of (4.37) and (4.38), first demonstrating that the ions have no pressure anisotropy.

4.4.2 Approximately Isotropic Ion Pressure

The result that ion guiding center motion does not lead to ion pressure anisotropy follows simply from (4.36): the $E_\parallel$ term is only present for electrons, and because of the small mass-ratio, only ion-ion self-collisions, which conserve momentum, enter $\mathcal{C}$. We therefore conclude that

$$\boldsymbol{b} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{p}_{bi} = 0 \tag{4.39}$$

Additionally, physical arguments can also show that ion anisotropy can be neglected due to ambipolarity requirements. Roughly speaking, a non-vanishing ion pressure anisotropy leads to radial ion flux that the electrons cannot match. To see how this comes about, consider again the exact divergence for a gyrotropic tensor, this time in terms of the scalar pressure, $p \equiv Tr(\boldsymbol{p})/3$, and anisotropy, $\Delta p \equiv p_\parallel - p_\perp$

$$\boldsymbol{\nabla} \cdot \boldsymbol{p}_{gt} = \boldsymbol{\nabla} p + \frac{1}{3}\left(\boldsymbol{\nabla}_\perp - 2\boldsymbol{\nabla}_\parallel\right)\Delta p - B\Delta p \nabla_\parallel \left(\frac{\boldsymbol{b}}{B}\right)$$

This divergence enhances the perpendicular flux

$$n\boldsymbol{V}_\perp \sim \frac{1}{m\Omega}\boldsymbol{b}\times\boldsymbol{\nabla}\cdot\boldsymbol{p}_{gt}$$

Clearly, the isotropic pressure leads to the familiar diamagnetic drift while the remaining terms arise from the presence of anisotropy.

Consider now which parts of the distribution function (2.52) can contribute to anisotropy

$$\Delta p = p_\parallel - p_\perp = \int d^3v\ m\left(u^2 - \frac{1}{2}v_\perp^2\right) f$$

Antisymmetry in velocity annihilate all parts of $\overline{f}$ except f_b. In the following chapter, we show that contributions to the stress tensor from $\widetilde{f}$ are second order in gyroradius. So the largest source, f_b, has at least one factor of the small gyroradius. The ion flux will therefore be proportional to ρ_i and the electron flux to ρ_e. The ratio of the two gyroradii goes as

$$\frac{\rho_i}{\rho_e} = \sqrt{\frac{m_i T_i}{m_e T_e}}.$$

Thus, given comparable temperatures, the electron flux will be too small by a factor of the square root of the mass-ratio. To prevent the unphysical appearance of a large charge separation due to this flux imbalance, the ions must attain a state of negligible anisotropy. Thus, $\Delta p_i = 0$, leading to (4.39) Following the discussions in [19] and above in Section 4.2.1, we say that the non-ambipolar flux due to time dependence has relaxed on the faster scale ν/ϵ leaving ambipolar flow and isotropic pressure for the ions on the drift time scale $\partial/\partial t \sim \delta\omega_t$.

Thus, ion bounce motion does not contribute to ion pressure anisotropy. However the energy-weighted collisional moment, (2.27)

$$\boldsymbol{G} \equiv \int d^3v\ m\boldsymbol{v}wf$$

is not small in the mass-ratio, so that bounce motion does contribute to the parallel divergence of the tensor $\boldsymbol{R}$. For electrons, anisotropy appears in both $\boldsymbol{P}$ and $\boldsymbol{R}$. We now attack in detail (4.37) and (4.38).

4.4.3 Anisotropy Calculation

4.4.3.1 Ions

Because of the small electron-ion mass-ratio, ion-ion collisions dominate the ion collision operator [14]. Additionally, the distribution function f^{loc} is localized in λ-space and therefore varies relatively sharply with λ near the trapped-untrapped boundary. The λ derivatives in the pitch-angle scattering part of the collision operator make the dominant term the pitch-angle scattering on the localized distribution

$$\mathcal{C}_i(f_{bi}) \rightarrow \mathcal{C}_{ii}(f_{bi}) \rightarrow \mathcal{C}_{\lambda i}\big(f_i^{loc}\big)$$

The pitch-angle collision operator

$$\mathcal{C}_{\lambda i} \equiv \nu_{\lambda ii}(\eta)\, \mathcal{L} \tag{4.40}$$

is made up of the pitch-angle scattering operator

$$\mathcal{L} \equiv 2h\sqrt{1-\frac{\lambda}{h}}\frac{\partial}{\partial\lambda}\lambda\sqrt{1-\frac{\lambda}{h}}\frac{\partial}{\partial\lambda} \tag{4.41}$$

and the energy dependent pitch-angle self-scattering collision frequency for species s

$$\nu_{\lambda ss}(\eta) \equiv \frac{3\sqrt{2\pi}}{\tau_s}\eta^{-5}\left[\eta \mathrm{erf}'(\eta) + \left(2\eta^2-1\right)\mathrm{erf}(\eta)\right] \tag{4.42}$$

written in terms of the normalized velocity

$$\eta \equiv \frac{v}{v_t} = \sqrt{\frac{mw}{T}}$$

and the the error function, erf(x), defined by its derivative

$$\mathrm{erf}'(x) \equiv \frac{2}{\sqrt{\pi}} e^{-x^2}$$

The calculation for $\boldsymbol{b} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{R}_{bi}$ requires the substitution of the localized distribution, (4.25), into $\mathcal{C}_{\lambda i}$; the resulting integrals are essentially the same as those required for the parallel flows computed in Section 4.3.4. The result is

$$\boldsymbol{b} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{R}_{bi} = \left(\frac{9}{4} - \alpha \right) \frac{\sqrt{2\epsilon} + h^2 - 1}{h} \frac{p_i T_i}{m_i \Omega_{pi} \tau_i} A_{2i}. \tag{4.43}$$

4.4.3.2 Electrons

The small mass-ratio allows several simplifications in the treatment of the ions which do not apply for electrons. Three new elements, in particular, modify the calculation for $\boldsymbol{b} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{P}_{be}$ and $\boldsymbol{b} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{R}_{be}$: the appearance of the parallel electric field in two places and the electron collision operator.

Regarding collisions, the electron operator must include interactions with both electrons and ions. The subtle point noted in [28] is that the form of the electron distribution function, $f_e = f_e^{non} + f_e^{loc}$, takes into account electron-electron collisions. In fact, equations (6.26) and (6.19) from this reference demonstrate that for the total distribution

$$\mathcal{C}_e(f_e) = (\nu_{\lambda ee} + \nu_{\lambda ei}) \, \mathcal{L} f_e^{loc} \tag{4.44}$$

where the collision frequency for electron-ion pitch-angle scattering is

$$\nu_{\lambda ei}(\eta) \equiv \frac{3\sqrt{\pi}}{4\tau_e} \eta^{-3}$$

In other words, the various collisional interactions between the Maxwellian, trapped, and untrapped electrons and ions are ultimately captured through the pitch-angle scattering of the trapped electrons off the ions and electron background.

Additionally, while the parallel electric field on the right-hand side of (4.12) can be neglected for the ions, it must be included in the electron version. The first consequence of this is the rather harmless appearance of $E_\parallel$ on the right-hand sides of (4.37) and (4.38); the resulting integrals are elementary, cf. (4.29). Perhaps more seriously, the localized electron distribution function, (4.26), includes the response to the parallel field: the Spitzer function, f_S. Because of rotational symmetry of the collision operator, f_S is a function only of the magnitude, η, and has been calculated numerically in the literature using the full collision operator [3, 32]. In evaluating (4.37) and (4.38), only the energy integrals are new, since

$$\mathcal{C}_\lambda(u^* f_S) = \nu_\lambda \mathcal{L}(u^* f_S) = \nu_\lambda f_S \mathcal{L} u^*$$

The resulting λ integrals are the same as for the ion case, and the energy integrals, requiring the moments

$$\int_0^\infty d\eta\, \eta^j \nu_\lambda e^{-\eta^2} f_S(\eta)$$

can be easily performed using the numerical data.

The final results for the parallel divergences are

$$\boldsymbol{b}\cdot\boldsymbol{\nabla}\cdot\boldsymbol{p}_{be} = -1.66\frac{z_1}{h}enE_\parallel - \frac{z_1}{h}\frac{n}{\Omega_{pe}\tau_e}\left[1.53\left(1+\frac{T_i}{T_e}\right)\frac{T_e}{n}\frac{\partial n}{\partial r} - 0.264\frac{\partial T_i}{\partial r} - 0.59\frac{\partial T_e}{\partial r}\right] \tag{4.45}$$

and

$$\boldsymbol{b}\cdot\boldsymbol{\nabla}\cdot\boldsymbol{R}_{be} = -2.97\frac{z_1}{h}\frac{ep_e}{m_e}E_\parallel - \frac{z_1}{h}\frac{p_e}{m_e}\frac{1}{\Omega_{pe}\tau_e}\left[1.71\left(1+\frac{T_i}{T_e}\right)\frac{T_e}{n}\frac{\partial n}{\partial r} - 0.294\frac{\partial T_i}{\partial r} + 1.04\frac{\partial T_e}{\partial r}\right] \tag{4.46}$$

where we abbreviate

$$z_1 \equiv \sqrt{2\epsilon} + h^2 - 1$$

Again, we point out that these are local expressions; when flux-surface averaged, however, they reproduce the neoclassical radial fluxes, as expected [16]

$$\left\langle nV^r \right\rangle \sim \left\langle \boldsymbol{B} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{P}_b \right\rangle$$

$$\left\langle Q^r \right\rangle \sim \left\langle \boldsymbol{B} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{R}_b \right\rangle$$

Chapter 5

Gyroviscosity

By gyroviscosity, we mean the off-diagonal elements of the pressure tensor driven by Larmor gyration in a magnetic field. Gyroviscosity can be treated simply and systematically in fluid equations, and is discussed throughout the literature, notably [1, 14, 26, 29]. The justification for constructing our own treatment is based on the guiding purpose of this work: to derive a fluid model that systematically includes the effects of gyration, guiding center motion, and parallel flows. In other words, we refrain from piecing together various previously derived results, and instead present the entire calculation consistently within the framework we have established throughout this thesis.

As in the previous chapter, the practical requirements are the non-scalar stresses (in particular the off-diagonal components) and their parallel divergences, to $\mathcal{O}(\delta^2)$, which will, as we shall see, provide additional neoclassical and parallel flow effects to the system of fluid equations they help close.

5.1 Magnetization Drives Stress

In Section 3.2, we alluded to the appearance of gyration driven stress components. these are higher order tensor analogues to the gyration driven perpendicular flows (2.38) and (2.37) and are demonstrated by the same means: small δ analysis of the fluid equations.

Gyroviscosity manifests in the stress evolution equation (2.22), which we now re-arrange as

$$\boldsymbol{b} \times \boldsymbol{P} + (\boldsymbol{b} \times \boldsymbol{P})^t = -\frac{1}{\Omega}\boldsymbol{S} \tag{5.1}$$

where we define the source

$$\boldsymbol{S} \equiv \frac{\partial \boldsymbol{P}}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{M}^{(3)} - \mathcal{Z}en\left(\boldsymbol{V}\boldsymbol{E} + \boldsymbol{E}\boldsymbol{V}\right) - m\boldsymbol{C}^{(2)} \tag{5.2}$$

The above operation of the symmetrized cross product has an exact algebraic inverse and the full solution is given by

$$\boldsymbol{P} = \boldsymbol{H}^{(2)} + \frac{1}{4\Omega}\boldsymbol{K}[\boldsymbol{S}] \tag{5.3}$$

where the symmetrized operator $\boldsymbol{K}$ is defined

$$\begin{aligned} \boldsymbol{K}\left[\boldsymbol{S}\right] &\equiv \left\|\boldsymbol{b} \times \boldsymbol{S} + 3\boldsymbol{b}\boldsymbol{b} \times (\boldsymbol{b} \cdot \boldsymbol{S})\right\| \\ &= \boldsymbol{b} \times \boldsymbol{S} + (\boldsymbol{b} \times \boldsymbol{S})^t + 3\boldsymbol{b}\boldsymbol{b} \times (\boldsymbol{b} \cdot \boldsymbol{S}) + 3\boldsymbol{b} \times (\boldsymbol{b} \cdot \boldsymbol{S})\,\boldsymbol{b} \end{aligned} \tag{5.4}$$

and the homogeneous solution $\boldsymbol{H}^{(2)}$ vanishes under the operation

$$\left\|\boldsymbol{b} \times \boldsymbol{H}^{(2)}\right\| = 0 \tag{5.5}$$

and is therefore gyrotropic.

Equations (5.3)-(5.5) are equivalent to (5.1) since they are only a re-arrangement. In particular, notice that the pressure tensor to be determined still appears on both sides of (5.3). The standard argument out of this circular reference is to notice that the largeness of the gyrofrequency in a magnetized plasma implies higher order corrections to the left hand side (stress tensor) due to lower order quantities on the right hand side. According to Section 2.4.3, closure of our model requires $\boldsymbol{P}$ to $\mathcal{O}(\delta^2)$, and thus, only first order accuracy is necessary in $\boldsymbol{S}$.

The expression for $\boldsymbol{S}$ can be simplified in the weakly collisional drift ordering context. Since temporal variation is a first order process,

$$\frac{\partial}{\partial t} \sim \delta\omega_t$$

the stress tensor need only be known to zeroth order, i.e. only the gyrotropic part $\boldsymbol{P}_0$. A fortunate algebraic property of (5.4) is that the operation exactly annihilates any gyrotropic tensor so that there will be no contributions to $\boldsymbol{P}$ from the time derivative term

$$\boldsymbol{K}\left[\frac{\partial \boldsymbol{P}_0}{\partial t}\right] = 0$$

The weak nature of collisional processes leads to further simplification. Consider the tensor moment that appears in (5.2),

$$\boldsymbol{\mathcal{C}}^{(2)} = \int d^3v \; m\boldsymbol{vv}\; \mathcal{C}(f)$$

The collision operator contains an implicit small factor of the collision frequency so that essentially only the zeroth order distribution function will be relevant. However, the only parts of the distribution function which survive the collision operator will be too small or have the wrong velocity symmetry to contribute to the integral. With these two considerations, (5.2) becomes

$$\boldsymbol{S} = \boldsymbol{\nabla}\cdot\boldsymbol{M}^{(3)} - \mathcal{Z}en\left(\boldsymbol{VE} + \boldsymbol{EV}\right) \tag{5.6}$$

The evolution of energy-weighted stress (2.25) can be analyzed in a similar manner to determine the second order corrections to $\boldsymbol{R}$ that are analogous to gyroviscosity, with the results

$$\boldsymbol{R} = \boldsymbol{H}^{(2)} + \frac{1}{4\Omega}\boldsymbol{K}[\boldsymbol{S}_R] \tag{5.7}$$

where, again, any gyrotropic tensor $\boldsymbol{H}^{(2)}$ can be added to the solution, and the source is

$$\boldsymbol{S}_R = \boldsymbol{\nabla} \cdot \frac{1}{2} Tr \boldsymbol{M}^{(5)} - \mathcal{Z}en \left(\boldsymbol{QE} + \boldsymbol{EQ} + \boldsymbol{E} \cdot \boldsymbol{M}^{(3)} \right) \tag{5.8}$$

5.2 Gyroviscosity Source

In (5.6) and (5.8), the only quantities that must be determined — in other words, that are not dynamical variables — are the third rank tensor moments, which we rename for notational convenience

$$\begin{aligned} \boldsymbol{N}_P &\equiv \boldsymbol{M}^{(3)} = \int d^3v \; m\boldsymbol{vvv} \, f \\ \boldsymbol{N}_R &\equiv \frac{1}{2} Tr \boldsymbol{M}^{(5)} = \int d^3v \; \boldsymbol{vvv} \frac{1}{2} m v^2 \, f \end{aligned} \tag{5.9}$$

and also introduce

$$\begin{aligned} \boldsymbol{Y}_P &\equiv \boldsymbol{VE} + \boldsymbol{EV} \\ \boldsymbol{Y}_R &\equiv \boldsymbol{QE} + \boldsymbol{EQ} + \boldsymbol{E} \cdot \boldsymbol{N}_P \end{aligned} \tag{5.10}$$

We are here only looking for off-diagonal components, since the inversion of (5.1) is powerless to specify the gyrotropic terms, and any gyrotropic terms here are smaller by δ than the anisotropy results of the previous chapter and are therefore safely ignorable, leaving

$$\boldsymbol{P}_i = \frac{1}{4\Omega} \boldsymbol{K} \left[\boldsymbol{\nabla} \cdot \boldsymbol{N}_i - \mathcal{Z}en \boldsymbol{Y}_i \right] \tag{5.11}$$

where $i = P, R$ indicates whether one is solving for $\boldsymbol{P}$ or $\boldsymbol{R}$. Once the stresses are calculated, one must take another divergence, since it is really only $\boldsymbol{b} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{P}$ and $\boldsymbol{b} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{R}$ that appear in the fluid evolution equations summarized in Section 2.4.3.

In fact, we see that, from the above ordering arguments, directly integrating (5.9) for $\boldsymbol{N}_i$ using a $\mathcal{O}(\delta)$ distribution function, such as (2.52), will produce the desired $\mathcal{O}(\delta^2)$ accuracy for the stress components. Notice that this simultaneously includes parallel particle flow, parallel heat flow, and bounce motion into the gyroviscosity calculation. Thus, here is the point where one includes other types of physics into the higher order gyration dynamics. When physical phenomena lead to spatial variation of the higher order energy flow tensors (5.9), the underlying process of gyration allows sampling of this variation and dictates corresponding momentum and heat fluxes. For the particular problem at hand, we are interested in what develops in toroidal plasmas, using (2.52) to determine the relevant forms for $\boldsymbol{N}_P$ and $\boldsymbol{N}_R$. From there, the divergence of (5.11) provides the final expressions to use in the fluid equations.

In the evaluation of (5.9), the velocity symmetries of the defining integrals and the various parts of the distribution function determine the precise tensor structure of the results. By definition, $\boldsymbol{N}_P$ is the third rank energy flux tensor whose trace is the total energy flux, cf. (2.6)

$$\boldsymbol{Q} \equiv \frac{1}{2} Tr \left(\boldsymbol{N}_P \right)$$

This vector establishes a direction of special significance for the structure of $\boldsymbol{N}_P$. In addition to being symmetric, $\boldsymbol{N}_P$ should sensibly be constructed from the relevant tensors, which are essentially only the rank two diagonal identity, $\boldsymbol{I}$, and the total energy flux, $\boldsymbol{Q}$. These considerations imply the form, confirmed upon integration of (5.9),

$$\boldsymbol{N}_P \sim \frac{2}{5} \| \boldsymbol{Q I} \| \tag{5.12}$$

which has components

$$\frac{2}{5} \left(Q_i \delta_{jk} + Q_k \delta_{ij} + Q_j \delta_{ik} \right)$$

Analogously, the structure of $\boldsymbol{N}_R$ is dictated by its vector trace and symmetry, resulting in a similar form

$$\boldsymbol{N}_R \sim \frac{2}{5}\|\boldsymbol{Q}_R \boldsymbol{I}\| \tag{5.13}$$

where evaluating the integrals in (5.9) shows

$$\boldsymbol{Q}_R \equiv \frac{7}{2}\frac{T}{m}\left(2\boldsymbol{q} + \frac{5}{2}p\boldsymbol{V}\right)$$

It it perhaps interesting to note that the total heat and particle fluxes manage to appear in (5.12) and (5.13). Indicating the contributions from each distribution function, the total result

$$\boldsymbol{N}_i[f] = \boldsymbol{N}_i[f_M] + \boldsymbol{N}_i[f_\|] + \boldsymbol{N}_i[\widetilde{f}] + \boldsymbol{N}_i[f_b]$$

shows that there will be an explicit neoclassical-type contribution from f_b, and since the flows in $f_\|$ are shifted by the neoclassical results, as discussed in Section 4.3.4, there is also a neoclassical-type contribution from $f_\|$. The form (4.32) was chosen to exactly preserve the identity of the total flows when taking the $\boldsymbol{v}$ and $\boldsymbol{v}v^2$ moments, but the contributions to $\boldsymbol{N}_i$ also combine into the total flows, despite the significant differences in velocity powers and velocity dependence between $f_\|$ and f_b. There are, however, a few other terms from f_b that we neglect to include; they are traceless and lead to stresses with zero parallel divergence, thus contributing neither to the total energy flux traces, nor to the parallel momentum or heat equations we are trying to close.

5.3 Gyroviscous Cancellation

The final requirement for the momentum evolution equation is the divergence of the stress tensor, with representative terms such as

$$\boldsymbol{\nabla} \cdot \boldsymbol{K}\left[\boldsymbol{\nabla} \cdot \left\{\left(\boldsymbol{q} + \frac{5}{2}p\boldsymbol{V}\right)\boldsymbol{I} + \boldsymbol{I}\left(\boldsymbol{q} + \frac{5}{2}p\boldsymbol{V}\right) + \dots\right\}\right]$$

Recall from (5.4) that $\boldsymbol{K}$ contains several $\boldsymbol{b}$ vectors, cross products, and symmetrizations; including the nested divergences, the above expression involves a fair bit of algebra. An interesting feature of the result, however, is that when inserted into (2.21), unexpected cancellations occur between parts of $d\boldsymbol{V}/dt$ and $\boldsymbol{\nabla}\cdot\boldsymbol{P}$[1], a circumstance known as the *gyro-viscous cancellation*[14, 26, 29]. To make this calculation more tractable, various approximations have been employed in the literature, motivating recent work establishing the correct general form, which unfortunately, appears quite complex[26]. Since our purpose here is instead to examine how the main physical effects of toroidal plasmas can be included systematically into a fluid model, we approximate to drastically simplify the calculation while still providing several interesting results: namely, that in the evaluation of the divergence of (5.11) we neglect derivatives of the magnetic field

$$\boldsymbol{\nabla}\boldsymbol{B} = 0 \tag{5.14}$$

The neglected terms merely come from repeated application of the product-rule for derivatives, so it is not a difficult matter to systematically retain them, if desired.

5.4 Gyroviscous Tensors

Performing the required integrals and applying $\boldsymbol{K}$ gives the following form for the gyroviscosity

$$\boldsymbol{\Pi}_{gv} = \boldsymbol{\Pi}_{\perp} + \boldsymbol{b}\boldsymbol{\Pi}_{\parallel} + \boldsymbol{\Pi}_{\parallel}\boldsymbol{b} \tag{5.15}$$

[1]These comments also apply to $\boldsymbol{\nabla}\cdot\boldsymbol{R}$, except it is a higher order cancellation in the heat flow equation.

with

$$
\begin{aligned}
\boldsymbol{\Pi}_\perp \equiv& \frac{1}{2\Omega}\left[p\boldsymbol{b}\times\boldsymbol{\nabla}\boldsymbol{V}_\perp + p\boldsymbol{\nabla}_\perp\boldsymbol{b}\times\boldsymbol{V} + \frac{2}{5}\boldsymbol{b}\times\boldsymbol{\nabla}\boldsymbol{q}_\perp + \frac{2}{5}\boldsymbol{\nabla}_\perp\boldsymbol{b}\times\boldsymbol{q}\right] \\
\boldsymbol{\Pi}_\parallel \equiv& \frac{1}{\Omega}\left[p\boldsymbol{b}\times\boldsymbol{\nabla}V_\parallel + p\nabla_\parallel\boldsymbol{b}\times\boldsymbol{V} + \frac{2}{5}\boldsymbol{b}\times\boldsymbol{\nabla}q_\parallel + \frac{2}{5}\nabla_\parallel\boldsymbol{b}\times\boldsymbol{q}\right] \\
&+ \frac{1}{\Omega}\left(\boldsymbol{b}\times\boldsymbol{V}\right)\left(\nabla_\parallel p - \mathcal{Z}enE_\parallel\right)
\end{aligned}
\tag{5.16}
$$

These expressions are consistent, given our simplifying assumptions, with previous studies of gyroviscosity[14, 26, 30].

The energy-weighted gyroviscosity has a similar form

$$
\boldsymbol{R}_{gv} = \boldsymbol{R}_\perp + \boldsymbol{b}\boldsymbol{R}_\parallel + \boldsymbol{R}_\parallel\boldsymbol{b} \tag{5.17}
$$

with

$$
\begin{aligned}
\boldsymbol{R}_\perp \equiv& \frac{1}{2\Omega}\left[\boldsymbol{b}\times\boldsymbol{\nabla}\boldsymbol{F}_\perp + \boldsymbol{\nabla}_\perp\boldsymbol{b}\times\boldsymbol{F} - \frac{7\mathcal{Z}e}{10m}\left\|\boldsymbol{b}\times\boldsymbol{E}\boldsymbol{Q}_\perp + \boldsymbol{b}\times\boldsymbol{Q}\boldsymbol{E}_\perp\right\|\right] \\
\boldsymbol{R}_\parallel \equiv& \frac{1}{\Omega}\left[\boldsymbol{b}\times\boldsymbol{\nabla}F_\parallel + \nabla_\parallel\boldsymbol{b}\times\boldsymbol{F} - \frac{7\mathcal{Z}e}{5m}\left(\boldsymbol{b}\times\boldsymbol{E}Q_\parallel + \boldsymbol{b}\times\boldsymbol{Q}E_\parallel\right)\right]
\end{aligned}
\tag{5.18}
$$

where we have abbreviated

$$
\boldsymbol{F} \equiv \frac{7}{2}\frac{T}{m}\left(p\boldsymbol{V} + \frac{4}{5}\boldsymbol{q}\right) \tag{5.19}
$$

From these we calculate the parallel divergences for the parallel flow evolution equations

$$
\begin{aligned}
\boldsymbol{b}\cdot\boldsymbol{\nabla}\cdot\boldsymbol{\Pi}_{gv} =& -mn\boldsymbol{V}_d\cdot\boldsymbol{\nabla}V_\parallel - \frac{\mathcal{Z}e}{\Omega}\boldsymbol{\nabla}\cdot\left(nE_\parallel\boldsymbol{b}\times\boldsymbol{V}\right) \\
&- \nabla_\parallel\left[\frac{p}{\Omega}\boldsymbol{b}\cdot\boldsymbol{\nabla}\times\boldsymbol{V}_\perp + \frac{2}{5\Omega}\boldsymbol{b}\cdot\boldsymbol{\nabla}\times\boldsymbol{q}_\perp + mn\boldsymbol{V}_d\cdot\boldsymbol{V}_\perp\right]
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{b}\cdot\boldsymbol{\nabla}\cdot\boldsymbol{R}_{gv} =& \frac{7}{5}\boldsymbol{V}_E\cdot\boldsymbol{\nabla}q_\parallel + \frac{7}{2}\boldsymbol{V}_E\cdot\boldsymbol{\nabla}\left(pV_\parallel\right) - \frac{7}{5}\frac{\mathcal{Z}e}{m\Omega}\boldsymbol{\nabla}\cdot\left(E_\parallel\boldsymbol{b}\times\boldsymbol{Q}\right) \\
&- \nabla_\parallel\left[\frac{7}{2}\frac{T}{m\Omega}\left(p\boldsymbol{b}\cdot\boldsymbol{\nabla}\times\boldsymbol{V}_\perp + \frac{4}{5}\boldsymbol{b}\cdot\boldsymbol{\nabla}\times\boldsymbol{q}_\perp\right)\right] \\
&- \nabla_\parallel\left[\frac{7}{2}p\boldsymbol{V}_d\cdot\boldsymbol{V}_\perp + \frac{7}{5}\boldsymbol{q}_\perp\cdot\boldsymbol{V}_\perp + \frac{28}{25}\frac{m}{pT}\boldsymbol{q}_\perp\cdot\boldsymbol{q}_\perp\right]
\end{aligned}
$$

As we have mentioned, we have taken the magnetic field to be uniform, cf. (5.14), to simplify the calculation of $\mathbf{\Pi}_{gv}$,$\boldsymbol{R}_{gv}$, and their divergences; the resulting expressions are still quite complicated, however. To again simplify, we recall that we have ordered parallel gradient smaller than perpendicular gradients, as in (1.3). This is intuitively reasonable and, as we shall show in Section 7.2, keeps interesting parallel gradient terms in the final equations, but does not require keeping the numerous parallel gradients in $\boldsymbol{b} \cdot \boldsymbol{\nabla} \cdot \mathbf{\Pi}_{gv}$ and $\boldsymbol{b} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{R}_{gv}$. In this case, we use the simpler forms

$$\begin{aligned}
\boldsymbol{b} \cdot \boldsymbol{\nabla} \cdot \mathbf{\Pi}_{gv} &= -mn\boldsymbol{V}_d \cdot \boldsymbol{\nabla} V_\parallel + \mathcal{O}\big(\delta^3\big) \\
\boldsymbol{b} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{R}_{gv} &= \frac{7}{5}\boldsymbol{V}_E \cdot \boldsymbol{\nabla} q_\parallel + \frac{7}{2}\boldsymbol{V}_E \cdot \boldsymbol{\nabla}\left(pV_\parallel\right) + \mathcal{O}\big(\delta^3\big)
\end{aligned} \tag{5.20}$$

where we also take advantage of the fact that $E_\parallel$ is not very large.

In summary, we have performed the algebraically intensive gyroviscosity calculation, specifically including the effects of parallel particle and heat flow and guiding center bounce motion. As we have noted, the neoclassical fluxes do not appear explicitly; the results are instead expressed in terms of the total fluxes, $\boldsymbol{V}$ and $\boldsymbol{q}$, notably including the parallel heat flux $q_\parallel$.

Chapter 6

Friction

The final closure requirements that must be addressed are the parallel frictions

$$F_\parallel \equiv \int d^3v \; mv_\parallel \, \mathcal{C}(f)$$

$$G_\parallel \equiv \int d^3v \; \frac{1}{2} m v^2 v_\parallel \, \mathcal{C}(f)$$

In an earlier work [5], we presented a simple method to calculate these in terms of the parallel flows, $V_\parallel$ and $q_\parallel$. Our original purpose was to demonstrate a simple relation between the perpendicular and parallel transport of particles and heat in a neoclassical plasma. However, the validity of some intermediate results, namely, the relation between parallel flows and parallel friction, rather than being restricted to the transport ordering of neoclassical theory, is perfectly applicable to the drift ordering of the present model. We therefore present that derivation of the expressions

$$F_\parallel = F_\parallel[nV_\parallel, q_\parallel]$$

$$G_\parallel = G_\parallel[nV_\parallel, q_\parallel]$$

which provide closure for the collisional moments.

The main point is that self-adjointness of the collision operator leads to a significant relation between the parallel moments of the distribution function f and $\mathcal{C}(f)$ that can be easily approximated.

6.1 Generalized Spitzer Functions

First, we combine the density and fluid velocity into the particle flux

$$\mathbf{\Gamma} \equiv n\boldsymbol{V} = \int d^3v\ \boldsymbol{v} f \tag{6.1}$$

The drift-ordered heat flux takes the form

$$\begin{aligned} \boldsymbol{q} &= \boldsymbol{Q} - \frac{5}{2} p \boldsymbol{V} \\ &= \boldsymbol{Q} - \frac{5}{2} T \mathbf{\Gamma} \\ &= T \int d^3v\ \left(\frac{mv^2}{2T} - \frac{5}{2} \right) \boldsymbol{v} f \end{aligned}$$

so that, in terms of the normalized velocity $\boldsymbol{\eta} = \boldsymbol{v}/v_t$, the parallel heat flux can be written

$$\frac{q_\parallel}{T} = v_t^4 \int d^3\eta\ \left(\eta^2 - \frac{5}{2} \right) \eta_\parallel f$$

Because of the analogous structure of the moments of f and moments of $\mathcal{C}(f)$, one can re-write them in terms of each other using replacements such as

$$\eta^j f \longleftrightarrow \eta^k\, \mathcal{C}(f)$$

This "conversion" between functions involving the collision operator brings to mind the generalized Spitzer problem. For example, defining the generalized Spitzer function f_{s2} as the solution to

$$\mathcal{C}(f_{s2}) = \eta_\parallel \left(\eta^2 - \frac{5}{2} \right) f_M \tag{6.2}$$

the heat flux can be re-written

$$\frac{q_\parallel}{T} = v_t^4 \int d^3\eta\ \mathcal{C}(f_{s2})\, \frac{f}{f_M}$$

Self-adjointness of the collision operator [14] implies that the argument of $\mathcal{C}$ can be switched, giving

$$\frac{q_\parallel}{T} = v_t^4 \int d^3\eta\ \frac{f_{s2}}{f_M}\, \mathcal{C}(f) \tag{6.3}$$

Although the Spitzer function f_{s2} essentially corresponds to the actual distribution function in the context of *collision-dominated* parallel transport, here it is interpreted simply as the function of η and $\eta_\parallel$ that is the solution to a particular mathematical equation, (6.2). Thus, expanding this function (but not the *actual* distribution function f) in velocity powers with coefficients a_n

$$f_{s2} = \eta_\parallel f_M \sum_n a_n \eta^{2n}$$

shows that $q_\parallel$ can be written as the sum

$$\frac{q_\parallel}{T} = v_t^4 \sum_n a_n \int d^3\eta \; \eta^{2n} \eta_\parallel \, \mathcal{C}(f) \tag{6.4}$$

where the integrals are, by definition, the parallel collisional moments. Similar arguments indicate that with another Spitzer function defined by

$$\mathcal{C}(f_{s1}) = \eta_\parallel f_M$$

the parallel particle flux can also be written as a different sum of parallel collisional moments. Higher fluid moments may be treated in the same way using Spitzer functions defined with appropriate inhomogeneous source terms.

These results depend only on the self-adjointness of the collision operator and smoothness of the Spitzer functions, and imply, fairly generally, that each parallel moment can be thought of as a certain linear combination of all the parallel collisional moments. Notice that to reach this conclusion, *it has not been necessary to say anything about the actual distribution function or the collisionality*; we have only converted moments of the unspecified quantity f into moments of the unspecified quantity $\mathcal{C}(f)$ using purely mathematical objects (generalized Spitzer functions).

6.2 Approximating the Spitzer Functions

However, while (6.3) is exact, without an expression for f_{s2} it is merely formal manipulation. A useful approximation is provided by the physical significance of the particle and heat flows: from (6.4), we notice that to the extent that the Spitzer functions f_{s1} and f_{s2} can be accurately represented using only the two lowest-order terms

$$f_{si} \sim \left(a_{i0} + a_{i1}\eta^2\right) \eta_{\|} f_M \tag{6.5}$$

there is a qualitatively correct expression relating $\Gamma_{\|}$ and $q_{\|}$ linearly to $F_{\|}$ and $G_{\|}$. Since the Spitzer functions are simply particular mathematical functions whose forms do not change, this result clearly holds in all collisionality regimes.

To determine the numerical coefficients that best represent the exact Spitzer functions f_{s1} and f_{s2}, we make use of a Laguerre decomposition. The associated Laguerre polynomials[1], $\{L_n\}$, form a complete orthogonal set such that any 'sufficiently' smooth function properly integrable over the infinite interval $(0, \infty)$ can be expressed as an infinite sum of the polynomials [22]. Since

$$\begin{aligned} L_0(x) &= 1 \\ L_1(x) &= \frac{5}{2} - x \end{aligned}$$

the approximate model Spitzer functions (6.5) are linear combinations of these lowest two polynomials with argument η^2. We choose the coefficients a_{ij} of the model functions to be the corresponding coefficients of the numerically calculated solutions [32]. In other words, our model functions have the same Laguerre components as the numerical solutions, for the

[1]We are referring to the Laguerre polynomials of order 3/2, but neglect to write the order for the sake of clarity.

first two terms, the higher order components of being neglected. Because of the orthogonality property of the polynomials, the coefficients of the numerical solutions are given by simple integrals of the numerical data with the proper weighting. Using these in (6.5), integration of (6.3) and its analog for $\Gamma_{\|}$ expresses the parallel flows as linear combinations of $F_{\|}$ and $G_{\|}$, or vice versa, through simple algebraic inversion.

6.3 Electrons and Ions

Technically, one must make the distinction between ion and electron versions of the Spitzer functions, even though the functions do not correspond to physical distributions in this context; the collision operators for the two species are different, afterall. The numerical data presented in Reference [32] was obtained using the *electron* collision operator, and therefore can only be used for electron quantities. Applying the above recipe results in

$$\begin{aligned} F_{\|e} &= \frac{m_e}{\tau_e}\left(-0.665\Gamma_{\|e} + 0.230\frac{q_{\|e}}{T_e}\right) \\ G_{\|e} &= \frac{T_e}{\tau_e}\left(-1.08\Gamma_{\|e} - 0.282\frac{q_{\|e}}{T_e}\right) \end{aligned} \tag{6.6}$$

Treatment of the ions requires an alternative approach. Fortunately, the consequences of the small electron-ion mass-ratio are significant simplifications. As noted in Section 4.4.3, to the lowest order in the mass-ratio, the ion collision operator includes only ion-ion collisions. Momentum conservation under purely self-collisions implies a vanishing friction force, so that

$$F_{\|i} \approx 0$$

To calculate $G_{\|i}$, notice that the above discussion demonstrated that $G_{\|i}$ will be proportional to the parallel flows, both $\Gamma_{\|i}$ and $q_{\|i}$. Dependence on $\Gamma_{\|i}$, however, implies a violation of Galilean invariance: any flow common to all ions should not affect collisions. We therefore expect

$$G_{\|i} = -\bar{\nu} q_{\|i} \tag{6.7}$$

where, rather than calculate $\bar{\nu}$, we choose the value so that when the neoclassical limit of the ion heat evolution equation, (2.63), is taken, the known neoclassical parallel heat flux, (4.31), is predicted. The point is that applying the transport ordering and flux-surface averaging (2.63) essentially eliminates all terms except $\boldsymbol{b} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{R}_{bi}$ and $G_{\|i}$; comparison of (4.43) and (4.31) makes the appearance of the neoclassical result seem likely. While we will discuss this in detail in Section 7.2, we note here, for concreteness, the reasonable result

$$\bar{\nu} \approx \frac{0.366}{\tau_i}$$

With (6.6)-(6.7), we have completed the task set out for this chapter: evaluation of the closure requirements $F_{\|e}$, $G_{\|e}$, and $G_{\|i}$ in terms of the dynamical variables, with special emphasis on the parallel flows.

Chapter 7

Synthesis

Having completed the calculations for the non-scalar stresses

$$\begin{aligned} \boldsymbol{p}^{ns} &= \boldsymbol{\Pi}_{gv} + \boldsymbol{P}_b \\ \boldsymbol{R}^{ns} &= \boldsymbol{R}_{gv} + \boldsymbol{R}_b \end{aligned} \tag{7.1}$$

and parallel collisional moments, we can now write the final system of equations. We will also examine the full system in the neoclassical limit.

7.1 Complete System

The evolution of the density and the two scalar pressures is given by

$$\begin{aligned} \frac{\partial n}{\partial t} + \boldsymbol{\nabla} \cdot (n\mathbf{V}) = 0 \\ \frac{3}{2}\left(\frac{\partial p_e}{\partial t} + \mathbf{V}_e \cdot \boldsymbol{\nabla} p_e\right) + \boldsymbol{\nabla} \cdot \mathbf{q}_e + \frac{5}{2} p_e \boldsymbol{\nabla} \cdot \mathbf{V}_e = 0 \\ \frac{3}{2}\left(\frac{\partial p_i}{\partial t} + \mathbf{V}_i \cdot \boldsymbol{\nabla} p_i\right) + \boldsymbol{\nabla} \cdot \mathbf{q}_i + \frac{5}{2} p_i \boldsymbol{\nabla} \cdot \mathbf{V}_i = 0 \end{aligned} \tag{7.2}$$

In the summed equation evolving ion parallel flow (2.65), we neglect electron non-scalar pressure and inertia as small in the mass ratio, and make use of the vanishing ion pressure anisotropy demonstrated in Section 4.4.2, leaving

$$m_i n \frac{\partial V_{\|i}}{\partial t} + m_i n \boldsymbol{V}_E \cdot \boldsymbol{\nabla} V_{\|i} + \nabla_{\|}\left(p_i + p_e\right) = 0 \tag{7.3}$$

For the perpendicular components of the momentum equation, we write the species sum of (2.62)

$$m_i n \frac{d\boldsymbol{V}_E}{dt} + \boldsymbol{\nabla}_\perp (p_i + p_e) + \left[m_i n \frac{d\boldsymbol{V}_{di}}{dt} + \boldsymbol{\nabla} \cdot \boldsymbol{\Pi}_{\perp i} \right] - \boldsymbol{J} \times \boldsymbol{B} = 0 \tag{7.4}$$

neglecting again the demonstrably small ion anisotropy and other electron mass terms. Employing the same approximations used in the parallel gyroviscous cancellation, we have

$$\left[mn \frac{d\boldsymbol{V}_d}{dt} + \boldsymbol{\nabla} \cdot \boldsymbol{\Pi}_\perp \right] = -\boldsymbol{\nabla}_\perp \chi_i \tag{7.5}$$

where

$$\chi \equiv \frac{p}{2\Omega} \boldsymbol{b} \cdot \boldsymbol{\nabla} \times \boldsymbol{V}_\perp + \frac{1}{5\Omega} \boldsymbol{b} \cdot \boldsymbol{\nabla} \times \boldsymbol{q}_\perp \tag{7.6}$$

is made up of the parallel vorticity and parallel heat vorticity. The final result becomes

$$m_i n \frac{\partial \boldsymbol{V}_E}{\partial t} + m_i n \boldsymbol{V}_i \cdot \boldsymbol{\nabla} \boldsymbol{V}_E + \boldsymbol{\nabla}_\perp (p_i + p_e - \chi_i) - \boldsymbol{J} \times \boldsymbol{B} = 0 \tag{7.7}$$

Since $\boldsymbol{J} \sim \boldsymbol{\nabla} \times \boldsymbol{B}$ and $\boldsymbol{V}_E \equiv \boldsymbol{E} \times \boldsymbol{B}/B^2$, we see that (7.7) is a closed equation that can be used to evolve the perpendicular components of the electric field.

Inserting the closure calculations for $F_{\| e}$ and $\boldsymbol{b} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{p}_{be}$ in to the "Ohm's law", (2.66), gives

$$J_\| - enV_{\| i} = \sigma_\star E_\| + J_b + J_p + J_q \tag{7.8}$$

where we combine the Spitzer conductivity with the neoclassical conductivity reduction as

$$\sigma_\star \equiv 1.96 \frac{ne^2 \tau_e}{m_e} \left(1 - 1.34 \frac{z_1}{h} \right) \tag{7.9}$$

and combine the various terms into "currents" for physical interpretation: the bootstrap current

$$J_b \equiv \frac{p_e}{B_p} \frac{z_1}{h} \left[-1.88 \left(1 + \frac{T_i}{T_e} \right) \frac{\partial \ln n}{\partial r} + 0.323 \frac{T_i}{T_e} \frac{\partial \ln T_i}{\partial r} - 0.177 \frac{\partial \ln T_e}{\partial r} \right] \tag{7.10}$$

the classical contribution from parallel gradients

$$J_p \equiv \frac{ep_e\tau_e}{m_e}\left(1.97\nabla_\parallel \ln p_e + 1.32\nabla_\parallel \ln T_e\right) \tag{7.11}$$

and a dynamical parallel heat current[1]

$$J_q \equiv 0.528\frac{e\tau_e}{T_e}\left(\frac{\partial q_{\parallel e}}{\partial t} + \boldsymbol{V}_E\cdot\boldsymbol{\nabla} q_{\parallel e}\right) \tag{7.12}$$

We can immediately see here that our system generalizes neoclassical theory by including terms annihilated by flux-surface averaging and by including the dynamical evolution of the parallel heat flow. To get a sense of the relative sizes of these currents, we compare them to the familiar bootstrap current: under our adopted orderings, both J_p and J_q can be significant, even substantially larger than J_b, depending on the particular development of parallel gradients and heat flow.

Turning now to electron parallel heat flow, equation (2.68) becomes

$$\frac{\partial q_{\parallel e}}{\partial t} + \boldsymbol{V}_E\cdot\boldsymbol{\nabla} q_{\parallel e} + \frac{5}{2}\frac{p_e}{m_e}\nabla_\parallel T_e + \Theta_\Delta - \Theta_c = 0 \tag{7.13}$$

where we define

$$\Theta_\Delta \equiv 1.18\frac{ep_e}{m_e}\frac{z_1}{h}E_\parallel - \frac{p_e}{m_e\Omega_p\tau_e}\frac{z_1}{h}\left[-2.12\left(1+\frac{T_i}{T_e}\right)\frac{T_e}{n}\frac{\partial n}{\partial r} + 0.366\frac{\partial T_i}{\partial r} + 2.51\frac{\partial T_e}{\partial r}\right] \tag{7.14}$$

and

$$\Theta_c \equiv G_{\parallel e} - \frac{5T_e}{2m_e}F_{\parallel e} = \frac{T_e}{\tau_e}\left(0.583nV_{\parallel e} - 0.857\frac{q_{\parallel e}}{T_e}\right) \tag{7.15}$$

Our use of the extended drift ordering $q_{\parallel e} \sim \mathcal{O}(1)$ simplified the electron gyroviscous heat cancellation; on the other hand, the effects of anisotropy and parallel frictions are obvious.

[1] To write the Ohm's law in closed form, it is necessary to re-write the heat current using the forthcoming equation for $\partial q_{\parallel e}/\partial t$.

An interesting feature of parallel heat flow evolution is that, through the higher-order "gyroviscous cancellation" from $\boldsymbol{b} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{R}_{gv}$, the diamagnetic drift velocity does not appear in the advective term

$$\boldsymbol{V} \cdot \boldsymbol{\nabla} \longrightarrow \boldsymbol{V}_E \cdot \boldsymbol{\nabla}$$

just as in the parallel momentum evolution equation.

For the ion version of (2.63), we have, after a number of cancellations,

$$\frac{\partial q_{\|i}}{\partial t} + \boldsymbol{V}_E \cdot \boldsymbol{\nabla} q_{\|i} + \frac{5}{2}\frac{p_i}{m_i}\nabla_{\|} T_i + \bar{\nu}\left(q_{\|i} - q_{\|bi}\right) - \frac{5}{2} p V_{\|} \boldsymbol{\nabla} \cdot \boldsymbol{V}_{di} = 0 \tag{7.16}$$

where we have used $\bar{\nu}$ to approximate the the calculation for $G_{\|i}$. In the next section we will determine its value by requiring that our system reproduce the neoclassical ion parallel heat flow in the neoclassical limit.

We also recall that the total first order fluxes are given by

$$\begin{aligned} \mathbf{V}_s &\equiv \mathbf{V}_{\perp s} + V_{\|s}\boldsymbol{b} \\ \mathbf{q}_s &\equiv \mathbf{q}_{\perp s} + q_{\|s}\boldsymbol{b} \end{aligned} \tag{7.17}$$

where the perpendicular fluxes

$$\begin{aligned} \mathbf{V}_\perp &= \mathbf{V}_E + \mathbf{V}_d \\ \mathbf{V}_E &= \frac{\mathcal{Z}e}{m\Omega}\mathbf{E} \times \mathbf{b} \\ \mathbf{V}_d &= \frac{1}{mn\Omega}\boldsymbol{b} \times \boldsymbol{\nabla} p \\ \mathbf{q}_\perp &= \frac{5}{2}\frac{p}{m\Omega}\boldsymbol{b} \times \boldsymbol{\nabla} T \end{aligned} \tag{7.18}$$

are non-dynamical, but $V_{\|}$ and $q_{\|}$ must be determined through the solution of the full system of equations. The current is related to the magnetic field through

$$\boldsymbol{J} = \frac{1}{\mu_0}\boldsymbol{\nabla} \times \boldsymbol{B}$$

and to evolve the magnetic field, we also must include Faraday's law

$$\frac{\partial \boldsymbol{B}}{\partial t} + \boldsymbol{\nabla} \times \boldsymbol{E} = 0 \tag{7.19}$$

7.2 Transport Reduction

The dynamics thus derived are more general than conventional treatments of transport and therefore describe a greater range of physical phenomena. While it is precisely this feature that is of primary importance, it is still useful to reproduce well-known transport results under the appropriate approximations. In this section, we consider the form of the equations under the transport ordering.

As discussed in Section 2.4.1, the essence of the transport ordering lies in the restriction to slow variation and flows,

$$\begin{aligned} \frac{\partial}{\partial t} &\sim \delta^2 \omega_t \\ V &\sim \delta v_t \\ q &\sim \delta p v_t \end{aligned} \tag{7.20}$$

and, importantly, the neglect of $\mathcal{O}(\delta^2)$ terms. Under these stricter constraints, many of the higher order corrections we have calculated vanish.

For example, consider the ion parallel heat flow evolution, (2.63), which becomes, after dropping $\mathcal{O}(\delta^2)$

$$\boldsymbol{b} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{R}_{bi} + \frac{5p}{2m} \nabla_{\|} T_i = G_{\| i}$$

or, equivalently, using (6.7) and noticing from (4.43) and (4.31) that in terms of the neoclassical parallel heat flow

$$\boldsymbol{b} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{R}_{bi} = -\frac{2}{5} \left(\frac{9}{4} - \alpha \right) \frac{q_{\| bi}}{\tau_i}$$

we finally have

$$-\frac{2}{5}\left(\frac{9}{4}-\alpha\right)\frac{1}{\tau_i}q_{\parallel bi}+\frac{5p}{2m}\nabla_\parallel T_i=-\bar{\nu}q_{\parallel i}$$

This expression now gives $q_\parallel$ directly, rather than through a differential equation. In addition to the transport ordering, neoclassical theory makes use of flux-surface averages, which would approximately annihilate the parallel gradient, showing that the judicious choice of

$$\bar{\nu}=\frac{2}{5}\left(\frac{9}{4}-\alpha\right)\tau_i\approx\frac{0.369}{\tau_i} \tag{7.21}$$

ensures our system returns the banana regime parallel heat flow (4.31) in the neoclassical limit[2].

We can also apply the transport ordering to the remaining evolution equations

$$\begin{aligned}
\boldsymbol{\nabla}\cdot n\boldsymbol{V}&=0\\
\boldsymbol{\nabla}\cdot\boldsymbol{Q}_i-\boldsymbol{V}_i\cdot\boldsymbol{\nabla}p_i&=0\\
\boldsymbol{\nabla}\cdot\boldsymbol{Q}_e-\boldsymbol{V}_e\cdot\boldsymbol{\nabla}p_e&=0\\
\nabla_\parallel\left(p_e+p_i\right)&=0\\
\boldsymbol{\nabla}_\perp\left(p_e+p_i\right)-\boldsymbol{J}\times\boldsymbol{B}&=0\\
\sigma_\star E_\parallel+enV_{\parallel i}+J_b+J_p-J_\parallel&=0\\
\frac{5}{2}\frac{p_i}{m_i}\nabla_\parallel T_i+\bar{\nu}\left(q_{\parallel i}-q_{\parallel bi}\right)&=0\\
\frac{5}{2}\frac{p_e}{m_e}\nabla_\parallel T_e+\Theta_\Delta+\Theta_c&=0
\end{aligned} \tag{7.22}$$

[2] Reference [4] contains a typographical error that underestimates the numerical coefficient by a factor of 2.

The flux-surface averaged form of these equations is

$$\begin{aligned}
\big\langle \boldsymbol{\nabla} \cdot n\boldsymbol{V} \big\rangle &= 0 \\
\big\langle \boldsymbol{\nabla} \cdot \boldsymbol{Q}_i \big\rangle &= \big\langle \boldsymbol{V}_i \cdot \boldsymbol{\nabla} p_i \big\rangle \\
\big\langle \boldsymbol{\nabla} \cdot \boldsymbol{Q}_e \big\rangle &= \big\langle \boldsymbol{V}_e \cdot \boldsymbol{\nabla} p_e \big\rangle \\
\big\langle \boldsymbol{\nabla}_\perp p_t \big\rangle &= \big\langle \boldsymbol{J} \times \boldsymbol{B} \big\rangle \\
\big\langle J_\| - enV_{\|i} \big\rangle &= \big\langle \sigma_\star E_\| + J_b \big\rangle \\
\big\langle q_{\|i} \big\rangle &= \big\langle q_{\|bi} \big\rangle \\
\big\langle \Theta_\Delta \big\rangle &= -\big\langle \Theta_c \big\rangle
\end{aligned} \tag{7.23}$$

where the define the total scalar pressure $p_t \equiv p_e + p_i$. For simplicity, we have also made use of the neoclassical result that parallel variation of p and T is small.

Notice that in this neoclassical limit, the current is made up of ion flow, the Ohmic current including the neoclassical conductivity reduction, and the bootstrap current, which is precisely the neoclassical result [16]. In addition, the flux-surface averaged parallel electron fluxes can be extracted from the above, using the "Ohm's law" and

$$\big\langle \Theta_\Delta \big\rangle = -\big\langle \Theta_c \big\rangle$$

In particular, our system gives a flux-surface averaged parallel electron velocity of

$$\begin{aligned}
\big\langle nV_{\|e} \big\rangle = {}& -1.96 \frac{e\tau_e}{m_e} \left(1 - 1.90\sqrt{\epsilon}\right) \big\langle nE_\| \big\rangle \\
& + \frac{p_e\sqrt{\epsilon}}{m_e \Omega_{pe}} \left\langle 2.66 \left(1 + \frac{T_i}{T_e}\right) \frac{\partial \ln n}{\partial r} - 0.457 \frac{T_i}{T_e} \frac{\partial \ln T_i}{\partial r} - 0.250 \frac{\partial \ln T_e}{\partial r} \right\rangle
\end{aligned} \tag{7.24}$$

which is the basis for the above bootstrap current. The result for the electron parallel heat flux is given by

$$\begin{aligned}
\big\langle q_{\|e}/T_e \big\rangle = {}& -1.34 \frac{e\tau_e}{m_e} \left(1 - 0.465\sqrt{\epsilon}\right) \big\langle nE_\| \big\rangle \\
& + \frac{p_e\sqrt{\epsilon}}{m_e \Omega_{pe}} \left\langle 1.65 \left(1 + \frac{T_i}{T_e}\right) \frac{\partial \ln n}{\partial r} + 0.289 \frac{T_i}{T_e} \frac{\partial \ln T_i}{\partial r} + 4.32 \frac{\partial \ln T_e}{\partial r} \right\rangle
\end{aligned} \tag{7.25}$$

We explicitly note again that the parallel fluxes and current are equal to the banana regime predictions only in a flux-surface averaged equilibrium sense — when the relaxation of $\partial/\partial t$ and averaging eliminates many terms. In general, the full system of coupled equations must be solved to determine the fluid quantities.

7.3 Conclusion

Returning to the full system of Section 7.1, we once again emphasize the features that make it an appropriate and useful model.

- Robust spatial variation

  We have preserved the full 3D spatial variation of the original fluid moment equations. Importantly, the fluid variables retain their variation along the magnetic field; the equations are not flux-surface averaged, and therefore provide completely local expressions and explicitly include parallel gradients such as $\nabla_\| p$ and $\nabla_\| T$.

- Finite Larmor radius effects

  In addition to the first-order perpendicular (diamagnetic) fluxes, our model retains higher order magnetization effects through the gyroviscosity tensors $\boldsymbol{P}_{gv}$ and $\boldsymbol{R}_{gv}$; perpendicular flows, parallel flows, and magnetic trapping are included in the calculation.

- Robust parallel fluxes

  Since we treat the weak collisionality (long mean-free-path) regime, we do not model $q_\|$ with only local gradients such as $\nabla_\| T$ but allow it to evolve consistently with the other dynamical variables.

- Neoclassical effects

  We have incorporated the magnetic trapping of guiding centers, resulting in local expressions for the parallel divergences of the stress tensors, $\boldsymbol{b} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{P}_b$ and $\boldsymbol{b} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{R}_b$, and the appearance of a local bootstrap current, J_b. Our model reproduces the principle predictions of neoclassical theory (conductivity reduction, bootstrap current, radial and parallel fluxes) in the appropriate limit, but has greater generality and applicability in its full form.

- Systematic ordering and derivation

  A focus on the process of fluid closure allowed a systematic derivation rather than conglomeration of previous results. The consistent use of the drift ordering makes the system appropriate for drift-wave instability analysis that requires a more comprehensive model for use with realistic geometries and fluxes.

In summary, we have constructed a system of 12 evolution equations — embodied in (7.2)- (7.19) — for the 12 dynamical variables

$$\{n, V_{\parallel i}, p_i, p_e, q_{\parallel i}, q_{\parallel e}, \boldsymbol{E}, \boldsymbol{B}\}$$

This coupled system includes the effects of gyration, magnetic trapping phenomena, and parallel particle and heat flux. The principle features of the derivation include: the emphasis on the significance of the parallel flows, the promotion of the parallel heat flux to a dynamical variable, the analogous treatment of gyromotion and guiding center motion, and the systematic adherence to the (extended) drift ordering. In the neoclassical limit, our system reproduces the radial and parallel neoclassical fluxes. However, the strength of our model lies its greater generality; it is tailored to situations where interesting temporal and spatial variation and fluxes coexist, producing a more dynamically active plasma.

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Vita

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